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Determination of Buckling Criteria by Minimization of Total Energy

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Abstract

With the assumption of constancy of the elastic constants in Hooke's Law over the range of strains involved, but no limitation upon magnitude of strains or displacements, formulae for strain energy are derived for several cases of interest. Exact buckling criteria are derived from the effect of small perturbations, from displacements in an unbuckled state, upon the total energy.

This method is applied to three problems: A column of rectangular section, compressed so as to keep its ends flat and parallel to their original position; a hollow circular cylinder, subjected to uniform pressure on its outer surface; and a hollow sphere, subjected to uniform external pressure. In all cases, for thin sections, results agree with those derived by other means. It is shown that with the given assumption, a buckling or critical load always exists, no matter what the geometric proportions. As an interesting sidelight, it is also shown that a finite pressure may expand the cylinder or sphere to infinity.

1. Introduction.

Assuming linearity between stress and strain when referred to principal directions, an exact, non-linear, formula for the strain energy may be derived in terms of displacements. Although not shown in this paper, the same formula results if Prof. Friedrichs's form of Hooke's Law for large displacements is utilized. The work done against external forces is also evaluated and added to give the total potential energy in the system. Buckling occurs when a small perturbation of displacements corresponding to unbuckled conditions results in a decrease of the total energy. Since the perturbation may be taken arbitrarily small, higher powers in the displacements corresponding to it and in derivatives of such displacements can be dropped without error leaving only quadratic terms to be considered, linear terms vanishing since the energy must be an extremum for the unbuckled state.

The change in energy due to the perturbation is now maximized by application of classical Calculus of Variations rules to yield linear differential equations determining optimal relations for the perturbation throughout the interior of the body. These are solved, subject to boundary conditions, in terms of arbitrary constants and the displacements obtained are substituted back into the expression for change in total energy. The limits of positive-definiteness of the energy with respect to the integration constants give the criteria for buckling.

2. The Strain Energy in Terms of Principal Strains.

If e_1, e_2, e_3 are the unit strains in the principal directions at any point in a strained body and p_1, p_2, p_3 are the corresponding stresses or forces per unit area, we assume that the latter are linear functions of the former so that, for isotropic material, we have

$$p_i = \lambda(e_1 + e_2 + e_3) + 2\mu e_i \ ; \quad i = 1,2,3$$

where λ and μ are elastic constants (Lamé). If w is the strain energy, we must also have

$$p_i = \frac{\partial w}{\partial e_i}$$

so that the energy must be given by

$$(1) \qquad 2w = \lambda(e_1 + e_2 + e_3)^2 + 2\mu(e_1^2 + e_2^2 + e_3^2)$$

if we assume it to be zero for zero strain.

If $p_3 = 0$, then

$$e_3 = - \frac{\lambda}{\lambda+2\mu}(e_1 + e_2)$$

and the above reduces to

$$(2) \qquad 2w = \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}(e_1 + e_2)^2 + \mu(e_1 - e_2)^2$$

If $e_3 = 0$, then we get the similar form

$$(3) \qquad 2w = (\lambda + \mu)(e_1 + e_2)^2 + \mu(e_1 - e_2)^2$$

3. Principal Strains in Terms of Displacement.

Consider a two-dimensional situation in Cartesian coordinates. A point with coordinates x,y moves to position x+u,y+v when the body is strained. An adjacent point an incremental distance dS away at angle α to the x axis has initial coordinates $x + dS \cos \alpha, y + dS \sin \alpha$ and, after strain, is dS_1 away at angle β with coordinates $x+u+dS_1 \cos \beta, y+v+dS_1 \sin \beta$. Assume these two points to determine one principal direction of strain (see Fig. 1). The second principal direction is at right angles to the first. A point on it an incremental distance dS from our first point has initial coordinates $x-dS \sin \alpha$, $y+dS \cos \alpha$. In the strained state, the principal directions remain perpendicular, but strains may differ so that the coordinates of the third point become $x+u-dS_2 \sin \beta$, $y+v+dS_2 \cos \beta$.

Since the points were taken very close to each other, continuity of displacement gives the relations:

$$dS_1 \cos \beta = (1 + u_x)dS \cos \alpha + u_y dS \sin \alpha$$

$$dS_1 \sin \beta = v_x dS \cos \alpha + (1 + v_y)dS \sin \alpha$$

$$dS_2 \sin \beta = (1 + u_x)dS \sin \alpha - u_y dS \cos \alpha$$

$$dS_2 \cos \beta = -v_x dS \sin \alpha + (1 + v_y)dS \cos \alpha$$

Eliminating α and β from these 4 equations gives the following 2 equations:

$$\left(\frac{dS_1}{dS}\right)^2 + \left(\frac{dS_2}{dS}\right)^2 = (1 + u_x)^2 + (1 + v_y)^2 + v_x^2 + u_y^2$$

$$\frac{dS_1}{dS} \cdot \frac{dS_2}{dS} = (1 + u_x)(1 + v_y) - u_y v_x$$

so that

$$(4)\qquad \left(\frac{dS_1}{dS} + \frac{dS_2}{dS}\right)^2 = (2 + e_1 + e_2)^2 = (2 + u_x + v_y)^2 + (u_y - v_x)^2$$

and

$$(5)\qquad \left(\frac{dS_1}{dS} - \frac{dS_2}{dS}\right)^2 = (e_1 - e_2)^2 = (u_x - v_y)^2 + (u_y + v_x)^2$$

In polar coordinates, a similar procedure can be followed (see Fig. 2) to give the continuity relations:

$$dS_1 \cos\beta = (1 + r_\rho)dS \cos\alpha - \frac{1}{\rho} r_\theta dS \sin\alpha$$

$$\frac{dS_1 \sin\beta}{\rho + r} = - \phi_\rho dS \cos\alpha + \frac{1}{\rho}(1 + \phi_\theta)dS \sin\alpha$$

$$dS_2 \sin\beta = (1 + r_\rho)dS \sin\alpha + \frac{1}{\rho} r_\theta dS \cos\alpha$$

$$\frac{dS_2 \cos\beta}{\rho + r} = \phi_\rho dS \sin\alpha + \frac{1}{\rho}(1 + \phi_\theta)dS \cos\alpha$$

and the strain equations

$$(6)\quad (2 + e_1 + e_2)^2 = [2 + (\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) + r_\rho]^2 + [(\rho + r)\phi_\rho - \frac{1}{\rho} r_\theta]^2$$

$$(7)\quad (e_1 - e_2)^2 = [(\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) - r_\rho]^2 + [(\rho + r)\phi_\rho + \frac{1}{\rho} r_\theta]^2$$

4. <u>The Rectangular Column</u>.

We take (Fig. 3) the origin at the mid-point of one edge with x in the direction of the load and y at right angles and assume zero stress in the third direction. We restrict any buckling so that the ends remain flat and parallel to their original position. For uniform compression, since there are no forces perpendicular to the column axis,

$$e_1 = -e$$

$$e_2 = e_3 = -\frac{\lambda e_1}{2(\lambda+\mu)} = \frac{\lambda e}{2(\lambda+\mu)}$$

so that

$$u = -ex$$

$$v = \frac{\lambda e}{2(\lambda+\mu)}\, y$$

If this is a stable condition, any slight change in displacements u and v must result in an increase in strain energy. Conversely, if there exist small changes in u and v which reduce the strain energy, the condition of uniform compression is unstable. We are thus led to assume

$$u = -ex + F(x,y)$$

$$v = \frac{\lambda e}{2(\lambda+\mu)}\, y + G(x,y)$$

and investigate the change in the total strain energy due to small F and G. Since we are considering no change in column length between the uniform compression and the buckled states, there is no work done against the external load that needs to be considered. We thus find

$$u_x = -e + F_x$$

$$u_y = F_y$$

$$v_x = G_x$$

$$v_y = \frac{\lambda}{2(\lambda+\mu)}\, e + G_y$$

We also note that our boundary conditions require F and G to vanish at $x = y = 0$ and that F is also zero for $x = 0$ or $x = L$. Substituting the above into equations (4) and (5) and the results

into (2), we obtain

$$2w = \frac{\mu(3\lambda+2\mu)}{\lambda + 2\mu}\left\{\sqrt{[2 - \frac{(\lambda+2\mu)e}{2(\lambda+\mu)} + F_x + G_y]^2 + (F_y - G_x)^2} - 2\right\}^2$$
$$+ \mu[F_x - G_y - \frac{(3\lambda+2\mu)}{2(\lambda+\mu)} e]^2 + \mu(F_y + G_x)^2$$

or

$$\frac{2w}{\mu} \approx \frac{(3\lambda+2\mu)}{\lambda + \mu} e^2 - \frac{2(3\lambda+2\mu)}{\lambda + \mu} eF_x + (F_x - G_y)^2 + (F_y + G_x)^2$$
$$+ \frac{(3\lambda+2\mu)}{\lambda+2\mu} (F_x + G_y)^2 - \frac{(3\lambda+2\mu)e}{4(\lambda+\mu) - (\lambda+2\mu)e} (F_y - G_x)^2$$

retaining terms only to second order in the derivatives of F and G. The total strain energy per unit width of column is

$$W = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^L w \, dxdy$$

Since

$$\int_0^L F_x \, dx = F \Big|_0^L = 0 ,$$

we have

$$\frac{2W}{\mu} = \frac{(3\lambda+2\mu)}{\lambda + \mu} hLe^2 + \frac{2\triangle W}{\mu}$$

where

$$\triangle W = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^L Q \, dxdy$$

and

$$Q = (F_x - G_y)^2 + (F_y + G_x)^2 + \frac{(3\lambda+2\mu)}{\lambda + 2\mu} (F_x + G_y)^2 - \frac{(3\lambda+2\mu)e}{4(\lambda+\mu)-(\lambda+2\mu)e}(F_y - G_x)^2$$

For $e < 0$, $Q > 0$ so that $\triangle W > 0$ and the uniform stress condition (tension, in this case) is stable as expected. Let us take

$$K = \frac{(3\lambda+2\mu)e}{4(\lambda+\mu)-(\lambda+2\mu)e}$$

$$M = \frac{3\lambda + 2\mu}{\lambda + 2\mu}$$

$$H = \frac{\pi h}{L}$$

and change variables to

$$\theta = \frac{\pi x}{L}$$

$$\eta = \frac{\pi y}{L}$$

then

$$\frac{L^2}{\pi^2} Q = (1 + M)(F_\theta^2 + G_\eta^2) + 2(M - 1)F_\theta G_\eta + (1 - K)(F_\eta^2 + G_\theta^2) + 2(1+K)F_\eta G_\theta$$

Since $F = 0$ at $\theta = 0$ or π, we may express it in the half-range Fourier series:

$$F = \sum F_n \sin n\theta$$

where the F_n are functions of η. We also note that $v_x = 0$ at $\theta = 0$ or π so that G may be developed in the half-range cosine series:

$$G = G_o + \sum G_n \cos n\theta$$

If we use primes to denote differentiation with respect to η, we thus get

$$F_\theta = \sum nF_n \cos n\theta$$

$$F_\eta = \sum F_n' \sin n\theta$$

$$G_\theta = - \sum nG_n \sin n\theta$$

$$G_\eta = G_o' + \sum G_n' \cos n\theta$$

Because of the orthogonality properties of the trigonometric

terms, we have

$$\int_0^L Q\,dx = \frac{L}{\pi}\int_0^\pi Q\,d\theta = \frac{\pi^2}{2L}\left(R_o + \sum R_n\right)$$

where

$$R_o = 2(1+M)G_o'^2$$

$$R_n = (1+M)(n^2F_n^2 + G_n'^2) + 2n(M-1)F_nG_n' + (1-K)(F_n'^2 + n^2G_n^2) - 2n(1+K)F_n'G_n$$

Since R_o is positive, we must have $G_o' = 0$ for minimum energy, which makes

$$G_o = -\sum G_n \Big|_{y=0}$$

For any n, $\triangle W$ is an extremum if we satisfy the conditions:

$$\frac{\partial R_n}{\partial F_n} = \left(\frac{\partial R_n}{\partial F_n'}\right)'$$

$$\frac{\partial R_n}{\partial G_n} = \left(\frac{\partial R_n}{\partial G_n'}\right)'$$

or when

$$n(K+M)G_n' = (1-K)F_n'' - (1+M)n^2F_n$$

and

$$-n(K+M)F_n' = (1+M)G_n'' - (1-K)n^2G_n$$

from which

$$F_n^{iv} - 2n^2F_n'' + n^2F_n = 0$$

or $F_n = A_n \sinh n\eta + B_n n\eta \cosh n\eta + C_n \cosh n\eta + D_n n\eta \sinh n\eta$

and, substituting this back into the equations:

$$G_n = \left[-A_n + \frac{(2+M-K)}{K+M} B_n\right]\cosh n\eta - B_n n\eta \sinh n\eta$$

$$+ \left[-C_n + \frac{(2+M-K)}{K+M} D_n\right]\sinh n\eta - D_n n\eta \cosh n\eta$$

Using the result of Appendix A, this makes

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} R_n \, d\eta = \left[(1-K)F_nF_n' + (1+M)G_nG_n' + n(M-K-2)F_nG_n\right] \Bigg|_{-\frac{H}{2}}^{\frac{H}{2}}$$

which upon substituting and simplifying yields:

$$\frac{1}{4nH}\int_{-\frac{H}{2}}^{\frac{H}{2}} R_n \, d\eta = A_n^2 \frac{\sinh nH}{nH} - A_nB_n\left[\frac{(2+M-K)}{M+K}\frac{\sinh nH}{nH} - \cosh nH\right]$$

$$+ B_n^2\left[\frac{(1+M)(1-K)(2+L-K)}{2(M+K)^2}\frac{\sinh nH}{nH} - \frac{(1+M)(1-K)}{2(M+K)}\cosh nH\right.$$

$$\left. + \frac{nH}{4}\sinh nH - \frac{(1+MK)}{M+K}\cosh^2\frac{nH}{2}\right]$$

$$+ C_n^2 \frac{\sinh nH}{nH} - C_nD_n\left[\frac{(2+M-K)}{M+K}\frac{\sinh nH}{nH} - \cosh nH\right]$$

$$+ D_n^2\left[\frac{(1+M)(1-K)(2+L-K)}{2(M+K)^2}\frac{\sinh nH}{nH} - \frac{(1+M)(1-K)}{2(M+K)}\cosh nH\right.$$

$$\left. + \frac{nH}{4}\sinh nH - \frac{(1+MK)}{M+K}\sinh^2\frac{nH}{2}\right]$$

Substituting the values of K and M:

$$\frac{1}{4nH}\int_{-\frac{H}{2}}^{\frac{H}{2}} R_n d\eta = A_n^2 \frac{\sinh nH}{nH} - A_nB_n\left\{\left[1 + \frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH} - \cosh n\right.$$

$$+ B_n^2\left\{\frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\left[1 + \frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH}\right.$$

$$\left. - \frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\cosh nH + \frac{nH}{4}\sinh nH - \frac{[(\lambda+2\mu)+2\lambda e]}{3\lambda+2\mu}\cosh^2\frac{nH}{2}\right\}$$

$$+ C_n^2 \frac{\sinh nH}{nH} - C_nD_n\left\{\left[1 + \frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH} - \cosh nH\right.$$

$$+ D_n^2\left\{\frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\left[1 + \frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH}\right.$$

$$- \frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\cosh nH + \frac{nH}{4}\sinh nH - \frac{[(\lambda+2\mu)+2\lambda e]}{3\lambda+2\mu}\sinh^2\frac{n}{2}$$

Examination shows that the terms in C_n and D_n are identical with those in A_n and B_n except for the last, in which $\sinh^2 \frac{nH}{2}$ appears instead of $\cosh^2 \frac{nH}{2}$. But

$$\cosh^2 \frac{nH}{2} = 1 + \sinh^2 \frac{nH}{2} > \sinh^2 \frac{nH}{2}$$

Hence any pair of values for A_n and B_n which make terms in them positive-definite will certainly make terms in C_n and D_n positive-definite and the former thus give the stronger criterion for stability. The condition for stability is, therefore:

$$4\,\frac{\sinh nH}{nH}\left\{\frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\left[1+\frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH} - \frac{2(\lambda+\mu)(1-e)}{3\lambda+2\mu}\cosh nH + \frac{nH}{4}\sinh nH - \frac{[(\lambda+2\mu)+2\lambda e]}{3\lambda+2\mu}\cosh^2\frac{nH}{2}\right\}$$
$$> \left\{\left[1+\frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH} - \cosh nH\right\}^2$$

for all n. This simplifies to:

$$\left[(1-2e)\frac{\sinh nH}{nH} - 1\right]\left\{\left[1+\frac{2(\lambda+2\mu)(1-e)}{3\lambda+2\mu}\right]\frac{\sinh nH}{nH} + 1\right\} > 0$$

which, since the right hand factor is positive, reduces to:

$$(1-2e)\frac{\sinh nH}{nH} - 1 > 0$$

or

$$e < \frac{1 - nH \cosh nH}{2}$$

which is most severe for minimum n, i.e. for n = 1, giving the limit of stability or critical strain as

$$e_{cr} = \frac{1 - H\,\mathrm{csch}\,H}{2}$$

$$< \frac{1}{2} \quad \text{for any } H\text{, no matter how large}$$

$$\approx \frac{H^2}{12} \quad \text{for } H^2 \ll 1\ .$$

Furthermore, $e_{cr} << 1$ implies $H^2 << 1$ so that the last approximation applies for small strain as well as small thickness compared to length of column.

The critical load is the load producing the critical strain and is

$$P_{cr} = \frac{\mu(3\lambda + 2\mu)}{\lambda+\mu} e_{cr} = Ee_{cr}$$

$$\approx \frac{EH^2}{12} = \frac{\pi^2 h^2 E}{12 L^2}$$

for small h/L or small strain in agreement with the usual Euler load for the assumed end conditions.

5. The Hollow Cylinder.

a. Symmetric Conditions.

In order to keep the problem two-dimensional, the cylinder will be assumed to be so loaded at its ends as to prevent any displacements parallel to its axis. Using polar coordinates, the principal strains in any cross-section are given by equations (6) and (7) which, substituted into equation (3) gives the strain energy density as:

$$(8) \quad 2w = (\lambda+\mu)\left\{\sqrt{[2 + (\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) + r_\rho]^2 + [(\rho+r)\phi_\rho - \frac{1}{\rho} r_\theta]^2} - 2\right\}^2$$
$$+ \mu\left\{[(\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) - r_\rho]^2 + [(\rho+r)\phi_\rho + \frac{1}{\rho} r_\theta]^2\right\}$$

with the total strain energy

$$(9) \qquad W = \iiint w \, dv = \int_b^a \int_a^{2\pi} w\rho \, d\rho d\theta \qquad \text{per unit length}$$

If deformation is symmetric, we have $\phi = r_\theta = 0$ so that

$$(10) \quad Q = 2w_\rho = (\lambda+\mu)\rho(\frac{r}{\rho} + r')^2 + \mu\rho(\frac{r}{\rho} - r')^2$$

$$= (\lambda + 2\mu)\frac{r^2}{\rho} + 2\lambda r r' + (\lambda+2\mu)\rho r'^2$$

where primes denote differentiation with respect to ρ.

Applying the usual Euler criterion, W is a minimum for

$$\frac{\partial Q}{\partial r} = (\frac{\partial Q}{\partial r'})'$$

or

$$\rho^2 r'' + \rho r' - r = 0$$

whose general solution is

$$(11) \qquad r = A\rho + \frac{B}{\rho}$$

The change in volume at the outer surface is, per unit length,

$$\pi(a + Aa + \frac{B}{a})^2 - \pi a^2$$

making the change in potential energy, or work done against the pressure p:

$$V = \pi p(Aa + \frac{B}{a})(2a + Aa + \frac{B}{a})$$

Also, substituting (11) into (10) and (9), the total strain energy is

$$W = 2\pi[(\lambda+\mu)A^2(a^2 - b^2) + \mu B^2(\frac{1}{b^2} - \frac{1}{a^2})]$$

so that the total energy change is

$$T = W + W' = \pi\Big\{2(\lambda+\mu)A^2(a^2-b^2)+2\mu B^2(\frac{1}{b^2} - \frac{1}{a^2})$$

$$+ p[2a(Aa + \frac{B}{a}) + (Aa + \frac{B}{a})^2]\Big\}$$

This is a minimum when

$$4(\lambda+\mu)(a^2 - b^2)A + p[2a^2 + 2a(Aa + \frac{B}{a})] = 0$$

and

$$4\mu(\frac{1}{b^2} - \frac{1}{a^2})B + p[2 + \frac{2}{a}(Aa + \frac{B}{a})] = 0$$

from which

(12) $$B = \frac{(\lambda + \mu)}{\mu} Ab^2$$

and

(13) $$p = - \frac{2(\lambda+\mu)(1 - \frac{b^2}{a^2})A}{1 + A + \frac{(\lambda + \mu)}{\mu} \frac{b^2}{a^2} A}$$

For small displacements, the latter approximates

$$p \approx -2(x+\mu)(1 - \frac{b^2}{a^2})A$$

in accord with linear theory. In (13), however, it is seen that, even for infinite displacement, P remains bounded. This probably has more meaning if b and a are interchanged so that p is an internal pressure, rather than external. Then we have the condition that the pressure

$$p = \frac{2(\lambda+\mu)(\frac{a^2}{b^2} - 1)}{1 + \frac{\lambda+\mu}{\mu} \frac{a^2}{b^2}}$$

causes infinite expansion. If the cylinder is thin compared to its mean radius, the maximum pressure approximates

$$p \approx \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} (\frac{a - b}{b})$$

b. Total Energy in Perturbed State.

We now assume small changes in displacement from that occurring under symmetric conditions and thus take

$$r = A\rho + \frac{B}{\rho} + F$$

where F is a function of ρ and θ, and permit ϕ to differ slightly from zero. Then (10) takes a more complex form in which, if powers above the second in F, ϕ, and their derivatives are dropped,

$$Q \approx 4\rho[(\lambda+\mu)A^2 + \frac{\mu B^2}{\rho^4}] + 4\rho[(\lambda+\mu)A + \frac{\mu B}{\rho^2}][\frac{F}{\rho} + (1 + A + \frac{B}{\rho^2})\phi_\theta]$$

$$+ 4\rho[(\lambda+\mu)A - \frac{\mu B}{\rho^2}]F_\rho + (\lambda + 2\mu)\rho(\frac{F^2}{\rho^2} + F_\rho^2) + 2\lambda FF_\rho$$

$$+ 2\lambda\rho(1 + A + \frac{B}{\rho^2})F_\rho\phi_\theta + 2[(\lambda+2\mu) + (3\lambda+4\mu)A + \frac{(\lambda+4\mu)B}{\rho^2}]F\phi_\theta$$

$$+ (\lambda+2\mu)\rho(1 + A + \frac{B}{\rho^2})^2\phi_\theta^2 + \frac{[\mu + (\lambda+2\mu)A]}{\rho(1+A)}[\rho^4(1 + A + \frac{B}{\rho^2})^2\phi_\rho^2 + F_\theta^2]$$

$$+ \frac{2(\mu - \lambda A)}{1 + A}\rho(1 + A + \frac{B}{\rho^2})\phi_\rho F_\theta$$

The work done against the external pressure also assumes a more complex form:

$$V = p\left\{\int_0^{2\pi} \frac{(\rho + r)^2}{2} d(\theta + \phi)\Big|_{\rho=a} - \pi a^2\right\}$$

$$= p\left\{\int_0^{2\pi} \frac{(a + Aa + \frac{B}{a} + F)^2}{2} d(\theta + \phi)\Big|_{\rho=a} - \pi a^2\right\}$$

$$= p\left\{\int_0^{2\pi} F(a + Aa + \frac{B}{a} + \frac{F}{2})d(\theta + \phi)\Big|_{\rho=a} + \pi(2a + Aa + \frac{B}{a})(Aa + \frac{B}{a})\right\}$$

But, to first order, for $\rho = a$,

$$d(\theta + \phi) = (1 + \phi_\theta)d\theta$$

Hence, to second order in F, ϕ, and derivatives

$$V = \pi p a^2(A + \frac{B}{a^2})(2 + A + \frac{B}{a^2}) + pa\int_0^{2\pi} F(1 + A + \frac{B}{a^2})d\theta$$

$$+ pa\int_0^{2\pi} F[\frac{F}{2} + a(1 + A + \frac{B}{a^2})\phi_\theta]d\theta$$

Terms in Q and V which are independent of F and ϕ correspond to the symmetric case and the total energy corresponding has already been minimized by our choice of A and B in accord with (12) and (13). Let us now turn our attention to terms in Q of first order in F and ϕ and their derivatives. Since

$$\int_0^{2\pi} \phi_\theta \, d\theta = \phi \Big|_0^{2\pi} = 0$$

For continuity, the term

$$4\rho[(\lambda + \mu)A + \frac{\mu B}{\rho^2}](1 + A + \frac{B}{\rho^2})\phi_\theta$$

does not contribute to the strain energy. Also

$$\frac{1}{2}\int_b^a \left\{4\rho[(\lambda + \mu)A - \frac{\mu B}{\rho^2}]F_\rho + \phi_\rho[(\lambda + \mu)A + \frac{\mu B}{\rho^2}]\frac{F}{\rho}\right\} d\rho$$

$$= 2\rho[(\lambda + \mu)A - \frac{\mu B}{\rho^2}]F\Big|_b^a$$

$$= 2(\lambda + \mu)Aa(1 - \frac{b^2}{a^2})F\Big|_{\rho=a} \quad \text{in view of (12)}$$

$$= -paF(1 + A + \frac{B}{a^2}) \quad \text{in view of (13)}$$

so that these terms in W cancel the first order term in V leaving only quadratic terms to be considered in the total energy. It is convenient, at this point, to introduce the general Fourier series expansions:

(14) $$F = F_o + \sum F_n \cos n\theta - \sum f_n \sin n\theta$$

and

(15) $$\phi = \xi_o + \frac{\sum \xi_n \cos n\theta + \sum \psi_n \sin n\theta}{\rho(1 + A + \frac{B}{\rho^2})}$$

where F_o, F_n, f_n, ξ_o, ξ_n, ψ_n are functions of ρ and independent of θ. Also, let

(16) $$K = \frac{\mu + (\lambda + 2\mu)A}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}(1 + A + \frac{B}{b^2}) > 0$$

if the center is not to close solid under symmetric loading. Substituting in W and V and noting orthogonality conditions for the Fourier terms, we find, using primes to denote differentiation with respect to ρ,

$$\Delta W = \frac{\pi}{2}\int_b^a R \, d\rho$$

where

$$R = R_o + \sum R_n + \sum S_n$$

with

$$R_o = 2(\lambda+2\mu)\frac{F_o^2}{\rho} + 2(\lambda+2\mu)\rho F_o'^2 + 4\lambda F_o F_o'$$

$$+ \frac{2(\lambda+2\mu)K}{1 + K}\rho^3(1 + A + \frac{B}{\rho^2})^2\xi_o'^2$$

$$R_n = (\lambda+2\mu)\left[\frac{F_n^2}{\rho} + \rho F_n'^2 + \frac{n^2\psi_n^2}{\rho} + \frac{2nF_n\psi_n}{\rho}\right] + 2\lambda F_n F_n' + 2\lambda n F_n'\psi_n$$

$$- 4\mu n F_n\psi_n' + \frac{(\lambda+2\mu)K}{1+K}\left[\frac{2n^2F_n^2}{\rho} + 2nF_n\psi_n' + \frac{2nF_n\psi_n}{\rho} + \rho\psi_n'^2\right.$$

$$\left. - \frac{2(1 + A - \frac{B}{\rho^2})}{1 + A + \frac{B}{\rho^2}}\psi_n\psi_n' + \frac{(1 + A - \frac{B}{\rho^2})^2}{(1 + A + \frac{B}{\rho^2})^2}\psi_n^2\right]$$

$$S_n = R_n(f_n, \xi_n)$$

and

$$\triangle V = p\,\frac{\pi}{2}\,[2F_o^2 + \sum(F_n^2 + 2nF_n\psi_n) + \sum(f_n^2 + 2nf_n\xi_n)]$$

Since the term in $\xi_o'^2$ is positive, W is minimized for $\xi_o' = 0$ or $\xi_o = 0$ if simple rotation is ignored. The terms in F_o and F_o' are independent of other functions and correspond to symmetric conditions previously treated. It is easy to verify that $F_o = 0$ gives minimum total energy as expected. Since terms in f_n and ξ_n are identical to those in F_n and ψ_n, we may consider them as merely corresponding to a rotation and limit consideration to the latter functions only. We note that functions of a particular subscript appear in separate terms from those of other subscripts so that the total energy consists of independent contributions by terms involving different subscripts, with each given by

$$T_n = \frac{\pi}{2}\left[\int_b^a R_n\, d\rho + p(F_n^2 + 2nF_n\psi_n)\right]$$

so that minimum energy corresponds to

$$\frac{\partial R_n}{\partial F_n} = \left(\frac{\partial R_n}{\partial F_n'}\right)'$$

and

$$\frac{\partial R_n}{\partial \psi_n} = \left(\frac{\partial R_n}{\partial \psi_n'}\right)'$$

which leads to the two differential equations:

$$(1+K)(\rho^2 F_n'' + \rho F_n' - F_n) - Kn^2 F_n = -n\rho\psi_n' + (1 + 2K)n\psi_n$$

and

$$K(\rho^2\psi_n'' + \rho\psi_n' - \psi_n) - (1 + K)n^2\psi_n = n\rho F_n' + (1 + 2K)nF_n$$

which are readily solved to give the general solutions:

$$F_n = C_n\rho^{n-1} + \frac{D_n}{\rho^{n-1}} + E_n\rho^{n+1} + \frac{H_n}{\rho^{n+1}}$$

$$\psi_n = -C_n\rho^{n+1} + \frac{(n-2) - 2K}{n + 2K}\,\frac{D_n}{\rho^{n-1}} - \frac{(n+2) + 2K}{n - 2K}\,E_n\rho^{n+1} - \frac{H_n}{\rho^{n+1}}$$

except for n = 1, for which we get

$$F_1 = C_1 + D_1 \log \rho + E_1\rho^2 + \frac{H_1}{\rho^2}$$

$$\psi_1 = -C_1 - \frac{D_1}{1+2K} - D_1 \log \rho - \frac{3 + 2K}{1 - 2K}\,E_1\rho^2 + \frac{H_1}{\rho^2}$$

Using the method given in Appendix A and noting that

$$\frac{2(\lambda + \mu)K}{1 + A + \frac{E}{\rho^2}} = 2\mu \qquad \text{at } \rho = b$$

$$= 2\mu - p \qquad \text{at } \rho = a$$

we find that

$$\frac{2T_n}{\pi} = \Big\{(\lambda+2\mu)[F_n(F_n + \rho F_n' + n\psi_n) + \frac{K}{1+K}\psi_n(\psi_n + \rho\psi_n' + nF_n)]$$

$$- 2\mu(F_n^2 + 2nF_n\psi_n + \psi_n^2)\Big\}\Big|_b^a + p(F_n^2 + 2nF_n\psi_n + \psi_n^2)\Big|_{\rho=a}$$

which becomes, upon substituting the above values for F_n and ψ_n,

$$(17) \quad \frac{2T_n}{\pi} = 2(n-1)C_n^2[2\mu(a^{2n-2} - b^{2n-2}) - pa^{2n-2}]$$

$$+ \frac{2(n-1)D_n^2}{(n+2K)^2 a^{2n-2} b^{2n-2}}[2\Big\{(1+2K)[2(\lambda+\mu)K - \mu] + \mu(n^2-1)\Big\}(a^{2n-2} - b^{2n-2})$$

$$+ p\Big\{(n^2-1) - (1+2K)^2\Big\}b^{2n-2}]$$

$$+ \frac{2(n+1)E_n^2}{(n-2K)^2}[2\Big\{(1+2K)[2(\lambda+\mu)K - \mu] + \mu(n^2-1)\Big\}(a^{2n+2} - b^{2n+2})$$

$$- p\Big\{(n^2-1) - (1+2K)^2\Big\}a^{2n+2}]$$

$$+ \frac{2(n+1)H_n^2}{a^{2n+2}b^{2n+2}}[2\mu(a^{2n+2} - b^{2n+2}) + pb^{2n+2}]$$

$$- \frac{4(n-1)(1+2K)}{n+2K}C_nD_np - \frac{4(n+1)(1+2K)}{n-2K}E_nH_np$$

$$+ \frac{4(n^2-1)}{n-2K}C_nE_n[2\mu(a^{2n} - b^{2n}) - pa^{2n}]$$

$$+ \frac{4(n^2-1)}{n+2K}\cdot\frac{D_nH_n}{a^{2n}b^{2n}}[2\mu(a^{2n} - b^{2n}) + pb^{2n}]$$

$$+ \frac{8(n^2-1)}{n^2-4K^2}D_nE_n[\Big\{2\mu(1+2K) - 2(\lambda+2\mu)K\Big\}(a^2 - b^2) - (1+2K)pa^2]$$

for $n \neq 1$, while

$$(18)\quad \frac{2T_1}{\pi} = \frac{D_1^2}{(1+2K)^2}\,[4(\lambda + 2\mu)K(1 + 2K)\log\frac{a}{b} + p]$$

$$+ \frac{4E_1^2(1+2K)}{(1-2K)^2}\,[\left\{4(\lambda+\mu)K - 2\mu\right\}(a^4-b^4) + (1+2K)pa^4]$$

$$+ \frac{4H_1^2}{a^4b^4}\,[2\mu(a^4 - b^4) + pb^4]$$

$$+ \frac{4D_1E_1}{1-4K^2}\,[\left\{4(\lambda+\mu)K - 2\mu\right\}(a^2-b^2) + (1+2K)pa^2]$$

$$- \frac{4D_1H_1}{(1+2K)a^2b^2}\,[2\mu(a^2 - b^2) + pb^2] - \frac{8(1+2K)}{1-2K}\,E_1H_1p$$

and is independent of C_1 as might be expected since terms in C_1 correspond merely to a translation without strain.

c. Stability of Symmetric State.

For the symmetric state to be stable, T_n must be positive for every n. For convenience, let us take

$$(19)\qquad k = \frac{b}{a} < 1$$

$$(20)\qquad s = 1 - \frac{p}{2\mu} = \frac{1 + A + \frac{B}{b^2}}{1 + A + \frac{B}{a^2}}$$

$$(21)\qquad t = \frac{2(\lambda+2\mu)K}{\mu(1+2K)} = \frac{2(\lambda+2\mu)(1 + A + \frac{B}{b^2})}{(\lambda+3\mu) + 2\mu(A + \frac{B}{b^2})}$$

all of which are positive if the center does not close solid during symmetric deformation.

Let us first consider T_1, given by (18). We substitute the new constants:

$$x_1 = -\frac{2H_1}{ab}$$

$$x_2 = \frac{2(1 + 2K)ab}{1 - 2K} E_1$$

$$x_3 = \frac{D_1}{1+2K}$$

and note that

$$\log \frac{a}{b} > \frac{a^2 - b^2}{a^2 + b^2}$$

to obtain

$$\frac{T_1}{\pi\mu} > x_1^2(k^{-2} - sk^2) + x_2^2[(t-s)k^{-2} - (t-1)k^2]$$

$$+ x_3^2[t(1 + 2T)^2 \frac{(k^{-1} - k)}{k^{-1} + k} + 1 - s]$$

$$+ 2x_1x_2(1-s) + 2x_1x_3(k^{-1}-sk) + 2x_2x_3[(t-s)k^{-1} - (t-1)k]$$

Now the quadratic form

$$\sum_i \sum_j a_{ij}x_ix_j \quad ; \qquad a_{ij} = a_{ji}$$

is positive definite if

$$\Delta_1 = a_{11} > 0$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0$$

etc. In the present case,

$$\triangle_1 = k^{-2} - sk^2 > 1 - sk^2 = \frac{(1 + A)(1 - k^2)}{1 + A + \frac{B}{a^2}} > 0$$

$$\triangle_2 = \begin{vmatrix} k^{-2}-sk^2 & 1 - s \\ 1 - s & (t-s)k^{-2}-(t-1)k^2 \end{vmatrix}$$

$$= [(t-s)k^{-2} - s(t-1)k^2][k^{-2} - k^2]$$

but

$$k^{-2} - k^2 > 0$$

and

$$(t-s)k^{-2} - s(t-1)k^2 = \frac{(1 + A + \frac{B}{b^2})M}{(1 + A + \frac{B}{a^2})[(\lambda+3\mu) + 2\mu(A + \frac{B}{b^2})]}$$

where

$$M = (\lambda+\mu)(1 + 2A + \frac{2B}{b^2})(k^{-2} - k^2) - 2(\lambda+2\mu)\frac{B}{b^2}(k^{-2} - 1)$$

$$= \frac{(\lambda+\mu)}{\mu}[\mu(k^{-2} - k^2) + 2(\lambda + 2\mu)A(1 - k^2)]$$

$$= \frac{(\lambda+\mu)}{\mu}[2(\lambda + \mu)K(1 - k^2) + \mu(k^{-2} - 2 + k^2)] > 0$$

Hence $\triangle_2 > 0$. Finally

$$\triangle_3 = \begin{vmatrix} k^{-2} - sk^2 & 1 - s & k^{-1} - sk \\ 1 - s & (t-s)k^{-2} - (t-1)k^2 & (t-s)k^{-1} - (t-1)k \\ k^{-1} - sk & (t-s)k^{-1} - (t-1)k & t(1+2K)^2\frac{(k^{-1}-k)}{k^{-1}+k} + 1 - s \end{vmatrix}$$

$$= \frac{1}{(k^{-1}+k)^2}\begin{vmatrix} k^{-2} - sk^2 & 1 - s & k^{-2} - sk^2 + 1 - s \\ 1 - s & (t-s)k^{-2} - (t-1)k^2 & 1-s+(t-s)k^{-2} - (t-1)k^2 \\ k^{-2}-sk^2+1-s & 1-s+(t-s)k^2 - (t-1)k^2 & t(1+2K)^2(k^{-2}-k^2) + (1-s)(k^{-1}+k)^2 \end{vmatrix}$$

$$= \frac{1}{(k^{-1} + k)^2} \begin{vmatrix} k^{-2} - sk^2 & 1 - s & 0 \\ 1 - s & (t-s)k^{-2} - (t-1)k^2 & 0 \\ 0 & 0 & L \end{vmatrix}$$

$$= \frac{\triangle_2 L}{(k^{-1} + k)^2}$$

where

$$L = t(k^{-2} - k^2)[(1 + 2K)^2 - 1] > 0$$

so that $\triangle_3 > 0$ and T_1 is positive definite and is a minimum for $D_1 = E_1 = H_1 = 0$.

We now turn to T_n, as given by (17), making the following substitutions, as well as those given by (19), (20), and (21):

$$\eta^2 = \frac{n^2 - 1}{(1+2K)^2} > 0$$

$$x_1 = \frac{2(n + 1)^{1/2}}{a^{\frac{n+1}{2}} b^{\frac{n+1}{2}}} H_n$$

$$x_2 = 2(n - 1)^{1/2} a^{\frac{n-1}{2}} b^{\frac{n-1}{2}} C_n$$

$$x_3 = \frac{2(1 + 2K)(n + 1)^{1/2}}{n - 2K} a^{\frac{n+1}{2}} b^{\frac{n+1}{2}} E_n$$

$$x_4 = \frac{2(1 + 2K)(n - 1)^{1/2}}{(n+2K)a^{\frac{n-1}{2}} b^{\frac{n-1}{2}}} D_n$$

to obtain

$$\frac{T_n}{\pi\mu} = x_1^2(k^{-n-1} - sk^{n+1}) + x_2^2(sk^{-n+1} - k^{n-1})$$

$$+ x_3^2[(t - s - s\eta^2)k^{-n-1} - (t - 1 + \eta^2)k^{n+1}]$$

$$+ x_4^2[(t - 1 + \eta^2)k^{-n+1} - (t - s + s\eta^2)k^{n-1}]$$

$$- 2x_1x_3(1 - s) - 2x_2x_4(1 - s)$$

$$+ 2x_1x_4\eta(k^{-n} - sk^n) + 2x_2x_3\eta(sk^{-n} - k^n)$$

$$- 2x_3x_4\eta[(t - 2s)k^{-1} - (t - 2)k]$$

Now

$$k^{-(n+1)} - sk^{n+1} > k^{-2} - sk^2 > 0 \qquad \text{as before.}$$

Further, an obvious criterion for stability is

$$sk^{-n+1} - k^{n-1} > 0 \qquad \text{for all } n > 1$$

so that

$$s > k^2$$

Assume this to hold and make the substitutions:

$$a_1 = k^{-n-1} - sk^{n+1} > 0$$

$$a_2 = sk^{-n+1} - k^{n-1} > 0$$

$$b_1 = k^{-n} - sk^n$$

$$b_2 = sk^{-n} - k^n$$

$$b_3 = (t - 2s)k^{-1} - (t - 2)k$$

$$c_1 = sk^{-n-1} - k^{n+1}$$

$$c_2 = k^{-n+1} - sk^{n-1}$$

$$d_1 = (t - s)k^{-n-1} - (t - 1)k^{n+1}$$

$$d_2 = (t - 1)k^{-n+1} - (t - s)k^{n-1}$$

Then

$$\frac{T_n}{\pi\mu} = a_1[x_1 - \frac{(1-s)}{a_1} x_3 + \eta \frac{b_1}{a_1} x_4]^2 + a_2[x_2 - \frac{(1-s)}{a_2} x_4 + \eta \frac{b_2}{a_2} x_3]^2$$

$$+ x_3^2[c_1\eta^2 + d_1 - \frac{(1-s)^2}{a_1} - \eta^2 \frac{b_2^2}{a_2}]$$

$$+ x_4^2[c_2\eta^2 + d_2 - \frac{(1-s)^2}{a_2} - \eta^2 \frac{b_1^2}{a_1}]$$

$$- 2\eta x_3 x_4 [b_3 - (1 - s)\frac{b_1}{a_1} - (1 - s)\frac{b_2}{a_2}]$$

and is positive definite only if

$$\eta^2[b_3 - (1-s)\frac{b_1}{a_1} - (1-s)\frac{b_2}{a_2}]^2$$

$$\leq [d_1 - \frac{(1-s)^2}{a_1} - \eta^2(\frac{b_2^2}{a_2} - c_1)][d_2 - \frac{(1-s)^2}{a_2} - \eta^2(\frac{b_1^2}{a_1} - c_1)]$$

or

$$\eta^2\Big\{[a_1b_3 - 2(1-s)b_1][a_2b_3 - 2(1-s)b_2) + (1-s)^2(a_1c_1 + a_2c_2 - 2b_1b_2)$$

$$+ a_2d_1(b_1^2 - a_1c_2) + a_1d_2(b_2^2 - a_2c_1)\Big\}$$

$$\leq [a_1d_1 - (1-s)^2][a_2d_2 - (1-s)^2] + \eta^4(b_1^2 - a_1c_2)(b_2^2 - a_2c_1)$$

or, inserting the values of the a's, b's, c's, and d's and factoring:

$$[\eta^2(k^{-1} - k)^2 - (k^{-2n} + k^{2n} - k^{-2} - k^2)]\Big\{\eta^2 s^2(k^{-1} - k)^2$$

$$- [s(t-1)(t-s)(k^{-2n} + k^{2n}) - (t-s)^2k^{-2} - s^2(t-1)^2k^2]\Big\} \geq 0$$

Now $$\frac{k^{-n} - k^n}{k^{-1}-k} = k^{-n+1} + k^{-n+3} + \ldots + k^{n-3} + k^{n-1} > n$$

Hence

$$\eta^2(k^{-1} - k)^2 - (k^{-2n} + k^{2n} - k^{-2} - k^2)$$

$$= \eta^2(k^{-1} - k)^2 - (k^{-n} - k^n)^2 + (k^{-1} - k)^2$$

$$< n^2(k^{-1} - k)^2 - (k^{-n} - k^n)^2 < 0$$

Hence stability requires the second factor, in the above, to be negative, or, rearranging it,

$$(n^2 - 1)(k^{-1} - k)^2 + [(\tfrac{t}{s} - 1)(1 + 2K)k^{-1} - (t - 1)(1 + 2K)k]^2$$

$$\leq (t - 1)(\tfrac{t}{s} - 1)(1 + 2K)^2(k^{-n} - k^n)^2$$

Substitution of values shows that

$$(\tfrac{t}{s} - 1)(1 + 2K)k^{-1} - (t - 1)(1 + 2K)k = k^{-1} - k$$

making the criterion

$$n^2(k^{-1} - k)^2 \leq (t - 1)(\tfrac{t}{s} - 1)(1 + 2K)^2(k^{-n} - k^n)^2$$

Since $(k^{-n}-k^n)/n$ increases with n, requirements are strongest for n = 2, for which

$$4(k^{-1} - k)^2 \leq (t - 1)(\tfrac{t}{s} - 1)(1 + 2K)^2(k^{-2} - k^2)^2$$

or

$$[1 + \frac{2(\lambda+2\mu)}{\mu} A][1 + \frac{2(\lambda+2\mu)}{\mu} Ak^2] \geq (\frac{2}{k^{-1} + k})^2$$

or

$$\Delta = 4(\frac{\lambda+2\mu}{\mu})^2k^2A^2 + 2(\frac{\lambda+2\mu}{\mu})(1 + k^2)A + (\frac{1 - k^2}{1 + k^2})^2 \geq 0$$

which is obviously satisfied for $A > 0$ ($p < 0$) and obviously not satisfied for

$$\frac{2(\lambda+2\mu)}{\mu} A = -1$$

corresponding to closure of the center to half its original size, since

$$b(1 + A + \frac{B}{b^2}) = b[1 + \frac{(\lambda + 2\mu)}{\mu} A]$$

Thus instability always occurs before compression is this large for any value of k. Furthermore, since s increases monotonically with A, we have, below this limit,

$$s - k^2 = \frac{\mu + (\lambda+2\mu)A}{\mu + [\mu + (\lambda+\mu)k^2]A} - k^2$$

$$> \frac{\lambda + 2\mu}{2\lambda + 3\mu - (\lambda + \mu)k^2} - k^2$$

$$= \frac{(1 - k^2)[\mu + (\lambda + \mu)(1 - k^2)]}{(\lambda + 2\mu) + (\lambda + \mu)(1 - k^2)} > 0$$

so that instability due to negative $\triangle$ always occurs at a lower deformation than that corresponding to $s \leq k^2$, and the limit of stability is given by the less negative root of the equation $\triangle = 0$, namely by

$$\frac{4(\lambda + 2\mu)}{\mu} k^2 A = -(1 + k^2) + \frac{\sqrt{1 + 14\, k^4 + k^8}}{1 + k^2}$$

For 1-k << 1, this reduces to

$$\frac{2(\lambda + 2\mu)}{\mu} A \approx - \frac{(1 - k)^2}{2}$$

and, from (13),

$$p \approx \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} (1 - k)^3$$

in agreement with other methods. We may also note, that $|A|$ is the average between radial and tangential strains and is, therefore, small when strains are small. Examination of $\triangle$ indicates

that instability can occur for small $|\Lambda|$ only if $1-k \ll 1$, hence the classical formula is valid for small strains.

6. The Hollow Sphere.

a. Conditions with Full Symmetry.

For axially-symmetric deformations, equations (6) and (7) are valid for the principal strains in a plane passing through the axis and the third principal direction is the tangent to a line of latitude. The change in length of a circle (see Fig. 5) coaxial with the assumed symmetry is in the ratio

$$\frac{dS_3}{dS} = \frac{(\rho + r)\sin(\theta + \phi)}{\rho \sin \theta}$$

so that the third principal strain is

$$e_3 = \frac{dS_3}{dS} - 1 = (1 + \frac{r}{\rho})(\cos\phi + \cot\theta \sin\phi) - 1$$

and the strain energy density (1) becomes

$$(22) \quad 2w = (\lambda+\mu)[2 + (\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) + r_\rho]^2 + (\lambda+\mu)[(\rho+r)\phi_\rho - \frac{1}{\rho} r_\theta]^2$$
$$+ [(\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) - r_\rho]^2 + \mu[(\rho+r)\phi_\rho + \frac{1}{\rho} r_\theta]^2$$
$$+ (\lambda + 2\mu)(1 + \frac{r}{\rho})^2(\cos\phi + \cot\theta \sin\phi)^2$$
$$- 2(3\lambda + 2\mu)(1 + \frac{r}{\rho})(\cos\phi + \cot\theta \sec\phi) + 3(3\lambda + 2\mu)$$
$$+ 2[\lambda(1 + \frac{r}{\rho})(\cos\phi + \cot\theta \sin\phi) - (3\lambda + 2\mu)]$$
$$\cdot \sqrt{[2 + (\frac{r}{\rho} + \phi_\theta + \frac{r}{\rho}\phi_\theta) + r_\rho]^2 + [(\rho + r)\phi_\rho - \frac{1}{\rho} r_\theta]^2}$$

and the total strain energy is

$$W = \iiint w\,dv = 2\pi \int_b^a \int_0^\pi w\rho^2 \sin\theta \,d\rho d\theta$$

For fully symmetric conditions, $\phi = r_\theta = 0$ and, using primes to denote differentiation with respect to ρ,

$$Q = 2w\rho^2 = 4(\lambda + \mu)r^2 + 4\lambda\rho r r' + (\lambda + 2\mu)\rho^2 r'^2$$

The strain energy is a minimum if

$$\frac{\partial Q}{\partial r} = \left(\frac{\partial Q}{\partial r'}\right)'$$

or

$$\rho^2 r'' + 2\rho r' - 2r = 0$$

whose general solution is

$$r = A\rho + \frac{B}{\rho^2}$$

which, substituted in the preceding, gives

$$W = 2\pi[(3\lambda + 2\mu)(a^3 - b^3)A^2 + 4\mu(\frac{1}{b^3} - \frac{1}{a^3})B^2]$$

The work done against the external pressure is the product of pressure and change of volume or

$$V = \frac{4}{3}\pi p a^3 [(1 + A + \frac{B}{a^3})^3 - 1]$$

making the total energy:

$$T = 2\pi\left\{(3\lambda + 2\mu)(a^3 - b^3)A^2 + 4\mu(\frac{1}{b^3} - \frac{1}{a^2})B^2 + \frac{2}{3}\,pa^3[(1 + A + \frac{B}{a^3})^3 - 1]\right\}$$

This is a minimum when

$$(3\lambda + 2\mu)(a^3 - b^3)A + pa^3(1 + A + \frac{B}{a^3})^2 = 0$$

and

$$4\mu(\frac{1}{b^3} - \frac{1}{a^3})B + p(1 + A + \frac{B}{a^3})^2 = 0$$

from which

(23) $$B = \frac{(3\lambda + 2\mu)}{4\mu} Ab^3$$

and

(24) $$p = - \frac{(3\lambda + 2\mu)A(1 - \frac{b^3}{a^3})}{[1 + A + \frac{(3\lambda+2\mu)}{4\mu} \cdot \frac{b^3}{a^3} A]^2}$$

Inspection of (24) shows that p increases with A, reaches a maximum, then decreases. This has more meaning if b and a are interchanged to obtain the case of internal pressure:

$$p = \frac{(3\lambda + 2\mu)A(\frac{a^3}{b^3} - 1)}{[1 + A + \frac{(3\lambda+2\mu)}{4\mu} \cdot \frac{a^3}{b^3} A]^2}$$

whose maximum is

$$p = \frac{(3\lambda + 2\mu)(\frac{a^3}{b^3} - 1)}{4 + \frac{3\lambda + 2\mu}{\mu} \cdot \frac{a^3}{b^3}}$$

and occurs when

$$A[1 + \frac{(3\lambda + 2\mu)}{4\mu} \frac{a^3}{b^3}] = 1$$

for which one easily verifies that the inside of the sphere has expanded to twice its original radius. Hence an internal pressure sufficient to expand a thin sphere to twice its dimensions will cause infinite expansion and in fact, less pressure is

required to maintain such expansion the further one goes in the process. This effect can be noted in blowing up a balloon.

b. Total Energy with Axially-Symmetric Perturbation.

We now consider small displacements from the fully-symmetric deformation by allowing non-zero values for ϕ and taking

$$r = A\rho + \frac{B}{\rho^2} + F$$

with ϕ and F functions of ρ and θ. For convenience, instead of ϕ itself, we take the new variable:

$$\psi = \rho(1 + A + \frac{B}{\rho^3})\phi$$

Substituting in the expression (22) for strain energy, we obtain to second order F, ψ, and their derivatives:

$$\begin{aligned}
2w\rho^2 = {} & 3[(3\lambda + 2\mu)A^2\rho^2 + 4\mu \frac{B^2}{\rho^4}] \\
& + 2\Big\{[(3\lambda + 2\mu)A\rho + 2\mu \frac{B}{\rho^2}][2F + \psi_\theta + \psi \cot \theta] \\
& + [(3\lambda + 2\mu)A\rho - 4\mu \frac{B}{\rho^2}]\rho F_\rho\Big\} \\
& + 4(\lambda + \mu)F^2 + 4\lambda\rho F F_\rho + (\lambda + 2\mu)\rho^2 F_\rho^2 + 2\lambda\rho F_\rho \psi_\theta \\
& + 2\lambda\rho F_\rho \psi \cot \theta + (\lambda + 2\mu)\psi_\theta^2 + 2\lambda\psi\psi_\theta \cot \theta \\
& + \frac{2[(\lambda + \mu) + (5\lambda + 4\mu)A + 2(\lambda + 2\mu)\frac{B}{\rho^3}]}{1 + A + \frac{B}{\rho^3}} F(\psi_\theta + \psi \cot \theta)
\end{aligned}$$

$$+ \frac{2(2\mu - 3\lambda A)}{2 + 2A - \frac{B}{\rho^3}} F_\theta [\rho\psi_\rho - \frac{(1 + A - \frac{2B}{\rho^3})}{1 + A + \frac{B}{\rho^3}} \psi]$$

$$+ \frac{[2\mu + (3\lambda + 4\mu)A + 2(\lambda - \mu)\frac{B}{\rho^3}]}{2 + 2A - \frac{B}{\rho^3}} \left\{ F_\theta^2 + [\rho\psi_\rho - \frac{(1 + A - \frac{2B}{\rho^3})}{1 + A + \frac{B}{\rho^3}} \psi]^2 \right\}$$

$$+ (\lambda + 2\mu)\psi^2 \cot^2\theta - \frac{[(3\lambda + 2\mu)A + 2\mu \frac{B}{\rho^3}]}{1 + A + \frac{B}{\rho^3}} \psi^2$$

Taking as element of volume, the conical shell with vertex angles θ and $\theta + d\theta$, we find the work done against external pressure as

$$V = \frac{2\pi}{3} p \int_0^\pi (a + Aa + \frac{B}{a^2} + F)^3 \sin(\theta + \phi) d(\theta + \phi) \Big|_{\rho = a} - \frac{4\pi}{3} pa^3$$

But, since $\phi = 0$ at $\theta = 0$ or π,

$$\frac{2\pi}{3} \int_0^\pi (a + Aa + \frac{B}{a^2})^2 \sin(\theta + \phi) \, d(\theta + \phi) = \frac{4\pi}{3} (a + Aa + \frac{B}{a^2})^3$$

Also, to first order,

$$\sin(\theta + \phi) \, d(\theta + \phi) \Big|_{\rho = a} = [\sin\theta + \frac{\psi \cos\theta + \psi_\theta \sin\theta}{\rho(1 + A + \frac{B}{\rho^3})}] d\theta \Big|_{\rho = a}$$

so that, noting the relation

$$\int_0^\pi F(\psi \cos\theta + \psi_\theta \sin\theta) d\theta = F\psi \sin\theta \Big|_0^\pi - \int_0^\pi F_\theta \psi \sin\theta \, d\theta$$

$$= - \int_0^\pi F_\theta \psi \sin\theta \, d\theta \, .$$

we have, to second order,

$$V = \frac{4\pi}{3} \rho a^3 [(1 + A + \frac{B}{a^3})^3 - 1]$$

$$+ 2\pi \rho a^2 (1 + A + \frac{B}{a^3})^2 \int_0^\pi F \sin \Theta \, d\Theta \Big|_{\rho=a}$$

$$+ 2\pi \rho a (1 + A + \frac{B}{a^3}) \int_0^\pi (F^2 - F_\Theta \psi) \sin \Theta \, d\Theta \Big|_{\rho=a}$$

The zero-order terms in W and V correspond to the fully-symmetric case already considered. We now turn to the first-order terms. We have

$$\int_0^\pi (\psi_\Theta + \psi \cot \Theta) \sin \Theta \, d\Theta = \psi \sin \Theta \Big|_0^\pi = 0$$

also

$$\int_b^a (\rho^2 F_\rho + 2\rho F) d\rho = \rho^2 F \Big|_b^a$$

and

$$\int_b^a (\frac{F_\rho}{\rho} - \frac{F}{\rho^2}) d\rho = \frac{F}{\rho} \Big|_b^a$$

so that

$$\int_b^a \left\{ 2[(3\lambda + 2\mu)A\rho + 2\mu \frac{B}{\rho^2}]F + [(3\lambda + 2\mu)A\rho - 4\mu \frac{B}{\rho^2}]\rho F_\rho \right\} d\rho$$

$$= [(3\lambda + 2\mu)A - 4\mu \frac{B}{\rho^3}]\rho^2 F \Big|_b^a$$

$$= [(3\lambda + 2\mu)A - 4\mu \frac{B}{a^3}]a^2 F \Big|_{\rho=a}$$

$$= (3\lambda + 2\mu)A(1 - \frac{b^3}{a^3})a^2 F \Big|_{\rho=a}$$

in view of (23). Comparing with (24), we thus see that first-order terms in W and V cancel as expected, leaving only the

quadratic terms in F and ψ contributing to change in total energy.

Considerations of continuity require that $\psi = F_\theta = 0$ at $\theta = 0$ or π. If we replace θ by the new independent variable

$$x = \cos\theta$$

and take

$$F = F(\rho, x)$$

and

$$\psi = \sin\theta\, G(\rho, x)$$

these conditions are satisfied. The previous formulae, retaining only second-order terms and inserting the value of p, then become:

$$\triangle V = -\frac{2\pi a\left(1 - \frac{b^3}{a^3}\right)(3\lambda + 2\mu)A}{1 + A + \frac{(3\lambda + 2\mu)}{4\mu}\cdot\frac{b^3}{a^3}A}\int_{-1}^{1}[F^2 + GF_x(1 - x^2)]dx\,\Big|_{\rho=a}$$

and

$$\triangle W = \pi\int_b^a\int_{-1}^{1} Q\, d\rho dx$$

where

$$Q = 4(\lambda + \mu)F^2 + 4\lambda\rho FF_\rho + (\lambda + 2\mu)\rho^2 F_\rho^2 + (\lambda + 2\mu)G^2$$

$$+ (\lambda + 2\mu)(1 - x^2)^2 G_x^2 - 2(\lambda + 2\mu)x(1 - x^2)GG_x + (\lambda + 2\mu)x^2G^2$$

$$+ 4\lambda\rho x F_\rho G - 2\lambda\rho(1 - x^2)F_\rho G_x - \frac{2(2\mu - 3\lambda A)}{2 + 2A - \frac{B}{\rho^3}}\rho(1 - x^2)F_x G_\rho$$

$$+ \left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right](1 - x^2)F_x^2 + \left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right]\rho^2(1 - x^2)G_\rho^2$$

$$+ 2\left[2(\lambda + 2\mu) - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right](1 - x^2)F_x G$$

$$-2\left[\frac{1 + A - \frac{2B}{\rho^3}}{1 + A + \frac{B}{\rho^3}}\right]\left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right]\rho(1 - x^2)GG_\rho$$

$$-\left\{2\mu\left[1 + \frac{\frac{3B}{\rho^3}\left(2 + 2A - \frac{B}{\rho^3}\right)}{\left(1 + A + \frac{B}{\rho^3}\right)^2}\right] - \frac{(2\mu - 3\lambda A)}{2 + 2A - \frac{B}{\rho^3}}\left[1 + \frac{\frac{3B}{\rho^3}\left(1+A - \frac{2B}{\rho^3}\right)}{\left(1 + A + \frac{B}{\rho^2}\right)^2}\right]\right\}(1-x^2)G^2$$

For minimum $\triangle W$,

$$\frac{\partial Q}{\partial F} = \frac{\partial}{\partial \rho}\left(\frac{\partial Q}{\partial F_\rho}\right) + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial F_x}\right)$$

and

$$\frac{\partial Q}{\partial G} = \frac{\partial}{\partial \rho}\left(\frac{\partial Q}{\partial G_\rho}\right) + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial G_x}\right)$$

or

(25) $$(\lambda+2\mu)(\rho^2F_{\rho\rho} + 2\rho F_\rho - 2F) - (\lambda+2\mu)\frac{\partial}{\partial x}\left[(1-x^2)(\rho G_\rho - G)\right]$$
$$+ \left[2\mu - \frac{2\mu-3\lambda A}{2+2A - \frac{B}{\rho^3}}\right]\frac{\partial}{\partial x}\left[(1-x^2)(F_x + \rho G_\rho + G)\right] = 0$$

and (after dividing through by $1-x^2$)

$$\left[2(\lambda+2\mu) - \frac{3(2\mu-3\lambda A)\frac{B}{\rho^3}}{\left(2 + 2A - \frac{B}{\rho^3}\right)^2}\right](F_x + G) + \left[\lambda + \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right]\rho F_{\rho x}$$

$$-(\lambda+2\mu)(1-x^2)G_{xx} + 4(\lambda+2\mu)xG_x - \left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right]\rho^2 G_{\rho\rho}$$

$$-\left[4\mu - \frac{2(2\mu - 3\lambda A)}{2 + 2A - \frac{B}{\rho^3}} + \frac{3(2\mu - 3\lambda A)\frac{B}{\rho^3}}{\left(2 + 2A - \frac{B}{\rho^3}\right)^2}\right]\rho G_\rho = 0$$

Since, however,

$$\rho \frac{\partial}{\partial \rho}\left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right] = \frac{3(2\mu - 3\lambda A)\frac{B}{\rho^3}}{(2 + 2A - \frac{B}{\rho^3})^2}$$

the latter may be written as

$$(26) \quad (\lambda+2\mu)[\rho F_{\rho x} + 2F_x - (1-x^2)G_{xx} + 4_x G_x + 2G] - \rho \frac{\partial}{\partial \rho}\left\{\left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right](F_x + \rho G_\rho + G)\right\} = 0$$

If we form the combination of these equations:

$$\frac{\partial}{\partial x}(1 - x^2)(26) + \rho \frac{\partial}{\partial \rho}(25)$$

we get, after dividing through by $\lambda+2\mu$:

$$\rho \frac{\partial}{\partial \rho}\left\{[\rho^2 F_{\rho\rho} + 2\rho F_\rho - 2F] - \frac{\partial}{\partial x}[(1-x^2)(\rho G_\rho - G)]\right\} + \frac{\partial}{\partial x}\left\{(1-x^2)[\rho F_{\rho x} + 2F_x - (1-x^2)G_{xx} + 4_x G_x + 2G]\right\} = 0$$

We now introduce a new function

$$(27) \qquad f = \rho F_\rho + 2F - \frac{\partial}{\partial x}[(1 - x^2)G]$$

and find that the last equation reduces to

$$\rho^2 f_{\rho\rho} + \frac{\partial}{\partial x}[(1 - x^2)f_x] = 0$$

whose general solution, since f is bounded at $x = \pm 1$, is

$$(28) \qquad f = \sum (C_n \rho^{n+1} + \frac{D_n}{\rho^n})P_n(x) \quad ; \qquad n = 0,1,2,\ldots$$

where C_n and D_n are arbitrary constants and $P_n(x)$ is the Legendre polynomial of order n.

If (27) and (28) be used to express G in terms of F and this be inserted in (25), we get the differential equation for F

$$\frac{\partial}{\partial x}[(1-x^2)F_x] + (\rho^2 F_{\rho\rho} + 4\rho F_\rho + 2F)$$

$$= \sum P_n(x)\Big\{[(n+2)C_n\rho^{n+1} - \frac{(n-1)D_n}{\rho^n}]$$

$$- \frac{(\lambda+2\mu)(2 + 2A - \frac{B}{\rho^3})}{2\mu + (3\lambda+4\mu)A - 2\mu\frac{B}{\rho^3}}[nC_n\rho^{n+1} - \frac{(n+1)D_n}{\rho^n}]\Big\}$$

which has the general solution

$$F = \sum H_n(p)P_n(x)$$

where

$$(29) \quad \rho^2 H_n'' + 4\rho H_n' - (n-1)(n+2)H_n = (n+2)C_n\rho^{n+1} - \frac{(n-1)D_n}{\rho^n}$$

$$- \frac{(\lambda+2\mu)(2 + 2A - \frac{B}{\rho^3})}{2\mu + (3\lambda+4\mu)A - \frac{2\mu B}{\rho^3}}[nC_n\rho^{n+1} - \frac{(n+1)D_n}{\rho^n}]$$

which has the solution

$$(30) \quad H_n = J_n\rho^{n-1} + \frac{K_n}{\rho^{n+2}} + \frac{\rho^{n+1}}{2n+3}C_n - \frac{D_n}{(2n-1)\rho^n}$$

$$- \frac{nC_n}{2n+1}\left[\rho^{n-1}\int_b^\rho \rho\xi\, d\rho - \frac{1}{\rho^{n+2}}\int_b^\rho \rho^{2n+2}\xi\, d\rho\right]$$

$$+ \frac{(n+1)D_n}{2n+1}\left[\rho^{n-1}\int_b^\rho \frac{\xi}{\rho^{2n}}\, d\rho - \frac{1}{\rho^{n+2}}\int_b^\rho \rho\xi\, d\rho\right]$$

in which J_n and K_n are arbitrary constants while

$$(31) \quad \xi = \frac{2(\lambda+\mu) - \lambda A - \frac{\lambda B}{\rho^3}}{2\mu + (3\lambda+4\mu)A - \frac{2\mu B}{\rho^3}} = \frac{\lambda}{2\mu} + \frac{(\lambda + 2\mu)}{2\mu}\cdot\frac{(1 + 4S)}{1 + S\frac{b^3}{\rho^3}}$$

where

(32) $$S = - \frac{(3\lambda + 2\mu)A}{2[2\mu + (3\lambda + 4\mu)A]}$$

From (27), we have

$$(1-x^2)G = \sum (\rho H_n' + 2H_n - C_n\rho^{n+1} - \frac{D_n}{\rho^n}) \int_{-1}^{x} P_n(x)\, dx \ ,$$

since G is bounded at x = -1 so that $(1-x^2)G = 0$ there. For $n = 0$, $P_n(x) = 1$ and boundedness of G at $x = 1$ means that

$$\rho H_o' + 2H_o - C_o\rho - D_o = 0$$

which, in turn, calls for $G = 0$ to correspond and reduces conditions to fully-symmetric ($\phi = 0$ and F independent of θ) for which the unperturbed deformation minimizes total energy so that $J_o = K_o = C_o = D_o = 0$ corresponds to minimum energy. For $n \neq 0$,

$$\int_{1}^{x} P_n(x)\, dx = - \frac{1}{n(n+1)} (1 - x^2)P_n'(x)$$

so that

$$G = \sum L_n(\rho)\, P_n'(x)$$

with

(33) $$L_n = - \frac{1}{n(n+1)} (\rho H_n' + 2H - C_n\rho^{n+1} - \frac{D_n}{\rho^n})$$

$$= \frac{-J_n\rho^{n-1}}{n} + \frac{K_n}{(n+1)\rho^{n+2}} + \frac{C_n\rho^{n+1}}{(n+1)(2n+3)} + \frac{D_n}{n(2n-1)\rho^n}$$

$$+ \frac{C_n}{2n+1}\left[\rho^{n-1}\int_b^\rho \rho\xi\, d\rho + \frac{n}{(n+1)\rho^{n+2}}\int_b^\rho \rho^{2n+2}\xi\, d\rho\right]$$

$$- \frac{D_n}{2n+1}\left[\frac{(n+1)}{n}\rho^{n-1}\int_b^\rho \frac{\xi}{\rho^{2n}}\, d\rho + \frac{1}{\rho^{n+2}}\int_b^\rho \rho\xi\, d\rho\right]$$

Noting that $\frac{\partial Q}{\partial F_x} = \frac{\partial Q}{\partial G_x} = 0$ at $x = \pm 1$, the final formula in

Appendix A is reduced to a single integral, given by

$$\triangle W = \pi \int_{-1}^{1} R\, dx \Big|_{b}^{a}$$

where

$$R = 2\lambda\rho F^2 + (\lambda+2\mu)\rho^2 FF_\rho + 2\lambda\rho x FG - \lambda\rho(1-x^2)FG_x$$

$$- \frac{(2\mu-3\lambda A)}{2+2A-\frac{B}{\rho^3}}\rho(1-x^2)F_x G + \left[2\mu - \frac{2\mu-3\lambda A}{2+2A-\frac{B}{\rho^3}}\right]\rho^2(1-x^2)GG_\rho$$

$$- \left[\frac{1+A-\frac{2B}{\rho^3}}{1+A+\frac{B}{\rho^3}}\right]\left[2\mu - \frac{2\mu-3\lambda A}{2+2A-\frac{B}{\rho^3}}\right]\rho(1-x^2)G^2$$

Now the Legendre polynomials have the following properties:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad \text{if } m \neq n$$

$$= \frac{2}{2n+1} \qquad \text{if } m = n$$

$$\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\,dx = 0 \qquad \text{if } m \neq n$$

$$= \frac{2n(n+1)}{2n+1} \qquad \text{if } m = n$$

$$\int_{-1}^{1} P_m(x)[(1-x^2)P_n''(x) - 2xP_n'(x)]dx = 0 \qquad \text{if } m \neq n$$

$$= -\frac{2n(n+1)}{2n+1} \qquad \text{if } m = n$$

so that both W and V split into independent portions, each corresponding to a single index n in the expansions for F and G with

$$\frac{2n+1}{2\pi} V_n = - \frac{2a(1 - \frac{b^3}{a^3})(3\lambda + 2\mu)A}{1 + A + \frac{(3\lambda+2\mu)}{4\mu}\cdot\frac{b^3}{a^3} A} [H_n^2 + n(n + 1)H_nL_n]\Big|_{\rho=a}$$

$$= \frac{8\mu Sa(1 - \frac{b^3}{a^3})}{1 + 2S - S \frac{b^3}{a^3}} [H_n^2 + n(n + 1)H_nL_n]\Big|_{\rho=a}$$

$$= \frac{8\mu S\rho(1 - \frac{b^3}{\rho^3})}{1 + 2S - S \frac{b^3}{\rho^3}} [H_n^2 + n(n + 1)H_nL_n]\Big|_b^a$$

and

$$\frac{2n+1}{2\pi} W_n = 2\lambda\rho H_n^2 + (\lambda+2\mu)\rho^2 H_nH_n' + \left[\lambda - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right] n(n + 1)\rho H_nL_n$$

$$+ \left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right] n(n + 1)\rho^2 L_nL_n'$$

$$- \left[\frac{1 + A - \frac{2B}{\rho^3}}{1 + A + \frac{B}{\rho^3}}\right]\left[2\mu - \frac{2\mu - 3\lambda A}{2 + 2A - \frac{B}{\rho^3}}\right] n(n + 1)\rho L_n^2 \Big|_b^a$$

However, from (33), we have

$$\rho H_n' = -n(n+1)L_n - 2H_n + C_n\rho^{n+1} + \frac{D_n}{\rho^n}$$

also, differentiating $\rho n(n+1)L_n$ given by the first line of (33) and eliminating the derivatives in H_n by use of (29), we get

$$\rho n(n+1)L_n' = -n(n + 1)L_n - n(n + 1)H_n$$

$$+ \frac{(\lambda + 2\mu)(2 + 2A - \frac{B}{\rho^3})}{2\mu + (3\lambda+4\mu)A - 2\mu \frac{B}{\rho^3}} \left[nC_n\rho^{n+1} - \frac{(n + 1)D_n}{\rho^n}\right]$$

so that

$$\frac{2n+1}{2\pi} W_n = \rho\left\{-4\mu H_n^2 - 4\mu n(n+1)H_nL_n - \left[\frac{2+2A-\frac{B}{\rho^3}}{1+A+\frac{B}{\rho^3}}\right]\left[2\mu - \frac{2\mu-3\lambda A}{2+2A-\frac{B}{\rho^3}}\right]n(n+1)L_n^2\right.$$

$$\left. + (\lambda+2\mu)H_n\left[C_n\rho^{n+1} + \frac{D_n}{\rho^n}\right] + (\lambda+2\mu)L_n\left[nC_n\rho^{n+1} - (n+1)\frac{D_n}{\rho^n}\right]\right\}\Bigg|_b^a$$

$$= \rho\left\{-4\mu[H_n^2 + n(n+1)H_nL_n] - \frac{2\mu(1+S\frac{b^3}{\rho^3})}{1+2S-S\frac{b^3}{\rho^3}} n(n+1)L_n^2\right.$$

$$\left. + (\lambda+2\mu)H_n\left[C_n\rho^{n+1} + \frac{D_n}{\rho^n}\right] + (\lambda+2\mu)L_n\left[nC_n\rho^{n+1} - (n+1)\frac{D_n}{\rho^n}\right]\right\}\Bigg|_b^a$$

and the corresponding total energy is

$$\frac{2n+1}{2\pi} T_n = -\frac{2\mu\rho(1+S\frac{b^3}{\rho^3})}{1+2S-S\frac{b^3}{\rho^3}}[2H_n^2 + 2n(n+1)H_nL_n + n(n+1)L_n^2]$$

$$+ (\lambda+2\mu)\rho\left\{H_n\left[C_n\rho^{n+1} + \frac{D_n}{\rho^n}\right] + L_n\left[nC_n\rho^{n+1} - (n+1)\frac{D_n}{\rho^n}\right]\right\}\Bigg|_b^a$$

We introduce the symbols:

(34) $$x = \frac{b}{\rho}$$

(35) $$M_o = \frac{2}{a^2}\int_b^a \rho\xi\, d\rho = 2k^2\int_k^1 \frac{\xi}{x^3}\,dx$$

(36) $$M_{n1} = \frac{2n+3}{a^{2n+3}}\int_b^a \rho^{2n+2}\xi\, d\rho = (2n+3)k^{2n+3}\int_k^1 \frac{\xi}{x^{2n+4}}\,dx$$

(37) $$M_{n2} = (2n-1)a^{2n-1}\int_b^a \frac{\xi}{\rho^{2n}}\,d\rho = (2n-1)k^{-2n+1}\int_k^1 \xi x^{2n-2}\,dx$$

which, using (30) and (33), makes the total energy

$$(38) \quad \frac{T_n}{4\pi\mu} = \frac{(n-1)}{n}\left[\frac{1+Sk^3}{1+2S-Sk^3} - k^{2n-1}\right] a^{2n-1} J_n^2$$

$$+ \frac{(n+2)}{n+1}\left[1 - \frac{(1+Sk^3)k^{2n+3}}{1+2S-Sk^3}\right]\frac{K_n^2}{b^{2n+3}}$$

$$+ \left\{\frac{(1+Sk^3)}{1+2S-Sk^3}\left[\frac{n(n-1)}{4(2n+1)^2} M_o^2 - \frac{(n+2)}{(n+1)(2n+3)^2}\left(1+\frac{nM_{n1}}{2n+1}\right)^2\right]\right.$$

$$\left. + \frac{(n+2)k^{2n+3}}{(n+1)(2n+3)^2} + \frac{(\lambda+2\mu)}{2\mu(n+1)(2n+3)}\left[1+\frac{nM_{n1}}{2n+1} - k^{2n+3}\right]\right\} a^{2n+3} C_n^2$$

$$+ \left\{\frac{(1+Sk^3)}{1+2S-Sk^3}\left[\frac{(n-1)}{n(2n-1)^2}\left(1-\frac{n+1}{2n+1}M_{n2}\right)^2 - \frac{(n+1)(n+2)}{4(2n+1)^2}M_o^2\right]k^{2n-1}\right.$$

$$\left. - \frac{(n-1)}{n(2n-1)^2} + \frac{(\lambda+2\mu)}{2\mu n(2n-1)}\left[1-k^{2n-1} - \frac{(n+1)}{2n+1}k^{2n-1}M_{n2}\right]\right\}\frac{D_n^2}{b^{2n-1}}$$

$$- \frac{(n-1)}{(2n+1)} \cdot \frac{(1+Sk^3)M_o}{(1+2S-Sk^3)} \cdot a^{2n+1} J_n C_n + \frac{(n+2)}{(2n+1)} \cdot \frac{(1+Sk^3)M_o}{(1+2S-Sk^3)} \cdot \frac{K_n D_n}{a^{2n+1}}$$

$$+ \frac{2(n-1)}{n(2n-1)}\left\{1 - \frac{(1+Sk^3)}{1+2S-Sk^3}\left[1 - \frac{(n+1)}{(2n+1)}M_{n2}\right]\right\} J_n D_n$$

$$+ \frac{2(n+2)}{(n+1)(2n+3)}\left\{1 - \frac{(1+Sk^3)}{1+2S-Sk^3}\left[1+\frac{nM_{n1}}{2n+1}\right]\right\} K_n C_n$$

$$+ \left\{\frac{(1+Sk^3)}{1+2S-Sk^3}\left[\frac{n-1}{2n-1} + \frac{n+2}{2n+3} + \frac{n(n+2)}{(2n+1)(2n+3)}M_{n1} - \frac{(n^2-1)}{4n^2-1}M_{n2}\right]\right.$$

$$\left. - \frac{\lambda+2\mu}{2\mu}\right\}\frac{M_o a^2 C_n D_n}{2n+1}$$

c. Axially-Symmetric Buckling Criteria.

We confine attention to compression under external pressure, that is, to $A < 0$.

Examining the expression for the n-th component of total energy (38), we see that terms involving J_n vanish for $n = 1$ as expected since this corresponds merely to sliding the sphere along the axis of symmetry.

For $n \geq 2$, the coefficient of J_n^2 must be positive if the fully-symmetric deformation is to be stable, so that

$$\frac{1 + Sk^3}{1 + 2S - Sk^3} > k^{2n-1}$$

This condition is strongest for lowest n, i.e. $n = 2$, giving one limit for stability as

$$\frac{1 + Sk^3}{1 + 2S - Sk^3} - k^3 = \frac{(1 - k^3)(1 - Sk^3)}{1 + 2S - Sk^3} > 0$$

or

$$k^{-3} > S > - \frac{1}{2 - k^3}$$

From (32), we see that the upper limit is reached for a lower absolute magnitude of A than needed to make S negative. We may thus take the criterion

(39) $$k^{-3} > S > 0 \quad ; \quad \text{if } A < 0$$

also, from (32). The upper limit corresponds to

$$A = - \frac{4\mu}{2(3\lambda + 4\mu) + (3\lambda + 2\mu)k^3}$$

for which the center radius is given by

$$b(1 + A + \frac{B}{b^3}) = b\left[1 + \frac{3(\lambda + 2\mu)}{4\mu} A\right]$$

$$= \frac{(3\lambda + 2\mu)(1 + k^3)b}{2(3\lambda + 4\mu) + (3\lambda + 2\mu)k^3} > \frac{b}{4}$$

so that instability is always reached before the external pressure is sufficient to reduce the inside radius to one fourth of its original value (lower bounds may be determined by later steps in the analysis).

Condition (39) suffices to insure that the coefficient of K_n^2 is also positive so that, within these limits, T_n is minimized if

$$\left[\frac{1 + Sk^3}{1 + 2S - Sk^3} - k^{2n-1}\right] J_n = \frac{n}{2(2n+1)} \cdot \frac{(1 + Sk^3)}{1 + 2S - Sk^3} a^2 M_o C_n$$

$$- \frac{1}{2n-1} \left\{1 - \frac{(1 + Sk^3)}{1 + 2S - Sk^3} \left[1 - \frac{(n+1)}{2n+1} M_{n2}\right]\right\} \frac{D_n}{a^{2n-1}}$$

and

$$\left[1 - \frac{(1 + Sk^3)k^{2n+3}}{1 + 2S - Sk^3}\right] K_n = - \frac{(n+1)}{2(2n+1)} \cdot \frac{(1 + Sk^3)}{1 + 2S - Sk^3} b^2 k^{2n+1} M_o D_n$$

$$- \frac{1}{2n+3} \left\{1 - \frac{(1 + Sk^3)}{1 + 2S - Sk^3} \left[1 + \frac{n}{2n+1} M_{n1}\right]\right\} b^{2n+3} C_n$$

For these values of J_n and K_n, the total energy becomes

$$\frac{T_n}{4\pi\mu} = \left[\frac{\beta_{n1}}{n+1} (\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1}) - \frac{n\alpha_{n2}k^{2n-1}M_o^2}{4(2n + 1)^2}\right] a^{2n+3} C_n^2$$

$$+ \left[\frac{\beta_{n2}}{n} (\frac{\lambda+2\mu}{2\mu} - \alpha_{n2}\beta_{n2}) - \frac{(n + 1)\alpha_{n1}k^{2n-1}M_o^2}{4(2n + 1)^2}\right] \frac{D_n^2}{b^{2n-1}}$$

$$+ [\alpha_{n1}\beta_{n1} + \alpha_{n2}\beta_{n2} - \frac{\lambda + 2\mu}{2\mu}] \frac{a^2 M_o C_n D_n}{(2n + 1)}$$

where

(40) $$\alpha_{n1} = \frac{(n + 2)(1 + Sk^3)}{(1 + 2S - Sk^3) - (1 + Sk^3)k^{2n+3}}$$

(41) $$\alpha_{n2} = \frac{(n - 1)(1 + Sk^3)}{(1 + Sk^3) - (1 + 2S - Sk^3)k^{2n-1}}$$

(42) $$\beta_{n1} = \frac{1 - k^{2n+3} + \frac{n}{2n+1} M_{n1}}{2n + 3}$$

(43) $$\beta_{n2} = \frac{1 - k^{2n-1} + \frac{(n+1)}{2n+1} k^{2n-1}M_{n2}}{2n - 1}$$

all of which are positive in view of (39).

In order for T_n to be positive definite, we must have

(44) $$\beta_{n1}\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1}\right) > \frac{n(n + 1)\alpha_{n2}k^{2n-1}M_o^2}{4(2n + 1)^2}$$

and

$$\left[\frac{\beta_{n1}}{n+1}\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1}\right) - \frac{n\alpha_{n2}k^{2n-1}M_o^2}{4(2n+1)^2}\right]\left[\frac{\beta_{n2}}{n}\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n2}\beta_{n2}\right) - \frac{(n+1)\alpha_{n1}k^{2n-1}M_o^2}{4(2n+1)^2}\right] > \left[\alpha_{n1}\beta_{n1}+\alpha_{n2}\beta_{n2} - \frac{\lambda+2\mu}{2\mu}\right]^2 \frac{k^{2n-1}M_o^2}{4(2n+1)^2}$$

or

(45) $$\left[\beta_{n1}\beta_{n2} - \frac{n(n+1)}{4(2n+1)^2} k^{2n-1}M_o^2\right]\left[\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1}\right)\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n2}\beta_{n2}\right) - \frac{n(n+1)}{4(2n+1)^2} \alpha_{n1}\alpha_{n2}k^{2n-1}M_o^2\right] >$$

But

$$\beta_{n1} > \frac{nM_{n1}}{(2n+1)(2n+3)} = \frac{nk^{2n+3}}{2n+1} \int_k^1 \frac{\xi}{x^{2n+4}}\, dx$$

and

$$\beta_{n2} > \frac{(n+1)k^{2n-1}M_{n2}}{(2n+1)(2n-1)} = \frac{(n+1)}{2n+1}\int_k^1 \xi x^{2n-2}\,dx$$

hence

$$\beta_{n1}\beta_{n2} > \frac{n(n+1)k^{2n+3}}{(2n+1)^2}\int_k^1 \frac{\xi}{x^{2n+4}}\,dx \int_k^1 \xi x^{2n-2}\,dx$$

Also, in view of (39), $\xi > 0$, so that

$$\int_k^1 \frac{\xi}{x^{2n+4}}\,dx \int_k^1 \xi x^{2n-2}\,dx = \int_k^1 \frac{\xi(x)}{x^{2n+4}}\,dx \int_k^1 \xi(y)y^{2n-2}\,dy$$

$$= \int_k^1\int_k^1 \frac{y^{2n-2}}{x^{2n+4}}\,\xi(x)\xi(y)\,dxdy$$

$$= \frac{1}{2}\int_k^1\int_k^1 \left(\frac{y^{2n-2}}{x^{2n+4}} + \frac{x^{2n-2}}{y^{2n+4}}\right)\xi(x)\xi(y)\,dxdy$$

$$> \int_k^1\int_k^1 \frac{1}{x^3y^3}\,\xi(x)\xi(y)\,dxdy$$

$$= \int_k^1 \frac{\xi(x)}{x^3}\,dx \int_k^1 \frac{\xi(y)}{y^3}\,dy = \left[\int_k^1 \frac{\xi}{x^3}\,dx\right]^2$$

making

$$\beta_{n1}\beta_{n2} > \frac{n(n+1)k^{2n-1}}{4(2n+1)^2}M_o^2$$

so that (45) reduces to

$$(46)\quad \left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1}\right)\left(\frac{\lambda+2\mu}{2\mu} - \alpha_{n2}\beta_{n2}\right) > \frac{n(n+1)}{4(2n+1)^2}\alpha_{n1}\alpha_{n2}k^{2n-1}M_o^2$$

Consider, at present, conditions for an almost solid sphere, i.e. $k \ll 1$. For this limit, we have the approximations:

$$\alpha_{n1} \approx \frac{n+2}{1+2S}$$

$$\alpha_{n2} \approx n-1$$

also

$$2k^2 \int_k^1 \frac{dx}{x^3} = (2n+3)k^{2n+3} \int_k^1 \frac{dx}{x^{2n+4}} = (2n-1) \int_k^1 x^{2n-2}\, dx \not\approx 1$$

and

$$2k^2 \int_k^1 \frac{dx}{x^3(1+Sx^3)} = 2k^2 \int_k^1 \frac{dx}{x^3} - 2k^2 \int_k^1 \frac{S\, dx}{1 + Sx^3} \approx 1$$

$$(2n+3)k^{2n+3} \int_k^1 \frac{dx}{x^{2n+4}(1+Sx^3)}$$

$$= (2n+3)k^{2n+3} \int_k^1 \frac{(1-Sx^3+S^2x^6-S^3x^9+\ldots+(-1)^m S^m x^{3m})dx}{x^{2n+4}}$$

$$- (2n + 3)k^{2n+3} \int_k^1 (-1)^m S^{m+1} x^{3m-2n-1}\, dx$$

$$\approx 1 \ , \quad \text{taking } 3m > 2n+1$$

However,

$$(2n-1) \int_k^1 \frac{x^{2n-2}\, dx}{1 + Sx^3} \approx (2n-1) \int_0^1 \frac{x^{2n-2}\, dx}{1 + Sx^3} = U_n$$

where

$$U_n = (2n-1) \int_0^1 \left[\frac{x^{2n-5}}{S} - \frac{x^{2n-8}}{S^2} + \frac{x^{2n-11}}{S^3} - \ldots + \frac{(-1)^m x^{2n+1-3m}}{S^{m-1}}\right] dx$$

$$- \frac{(-1)^m(2n - 1)}{S^{m-1}} \int_0^1 \frac{x^{2n+1-3m}}{1 + Sx^3}\, dx$$

with

$$3m = 2n + 1 \ , \ 2n \ , \text{ or } 2n-1$$

so that

$$(47) \quad U_n = \frac{(2n-1)}{3} \left[\frac{3}{(2n-4)S} - \frac{3}{(2n-7)S^2} + \frac{3}{(2n-10)S^3} - \ldots - \frac{1}{S^{(2n-4)/3}}\right]$$

$$+ \frac{(2n - 1)}{3S^{(2n-1)/3}} \log\,(1+S) \quad \text{for } 2n-1 = 3m$$

$$= (2n-1) \left[\frac{1}{(2n-4)S} - \frac{1}{(2n-7)S^2} + \frac{1}{(2n-10)S^3} - \ldots - \frac{1}{2S^{(2n-3)/3}}\right]$$

$$+ \frac{(2n-1)}{S^{(2n-1)/3}} \left[\frac{1}{6} \log \frac{(1+S^{1/3})^3}{(1 + S)} - \frac{1}{\sqrt{3}} \arctan \frac{\sqrt{3}\, S^{1/3}}{(2-S^{1/3})}\right] , \quad 2n=3m$$

$$= (2n-1)\left[\frac{1}{(2n-4)S} - \frac{1}{(2n-7)S^2} + \frac{1}{(2n-10)S^3} - \cdots + \frac{1}{S^{(2n-2)/3}}\right]$$

$$+ \frac{(2n-1)}{S^{(2n-1)/3}}\left[\frac{1}{6}\log\frac{(1+S^{1/3})^3}{(1+S)} + \frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}\,S^{1/3}}{2-S^{1/3}}\right]$$

for $2n+1 = 3m$

Hence

$$\beta_{n1} \approx \frac{1}{2n+3} + \frac{n}{(2n+1)(2n+3)}\left[\frac{\lambda}{2\mu} + \frac{(\lambda + 2\mu)(1 + 4S)}{2\mu}\right]$$

$$\beta_{n2} \approx \frac{1}{2n-1} + \frac{(n+1)}{(4n^2-1)}\left[\frac{\lambda}{2\mu} + \frac{(\lambda + 2\mu)(1 + 4S)}{2\mu}\,U_n\right]$$

also

$$M_o \approx \frac{\lambda}{2\mu} + \frac{(\lambda + 2\mu)(1 + 4S)}{2\mu}$$

and

$$k^{2n-1}M_o^2 \approx 0$$

reducing conditions (44) and (46) to

$$\frac{\lambda + 2\mu}{2\mu} > \alpha_{n1}\beta_{n1}$$

and

$$\frac{\lambda + 2\mu}{2\mu} > \alpha_{n2}\beta_{n2}$$

Now

$$\frac{\lambda+2\mu}{2\mu} - \alpha_{n1}\beta_{n1} \approx \frac{(\lambda+2\mu)}{2\mu}\left[1 - \frac{n(n+2)(1+4S)}{(2n + 1)(2n + 3)(1 + 2S)}\right]$$

$$- \frac{(n+2)[n\lambda + (2n+1)2\mu]}{2\mu(2n+1)(2n+3)(1+2S)}$$

$$> \frac{(\lambda+2\mu)}{2\mu}\left[1 - \frac{(n+2)(2n + 1 + n + 4nS)}{(2n+1)(2n+3)(1+2S)}\right]$$

$$= (\lambda+2\mu)[(n^2+n+1) + 2(2n^2+4n+3)S] > 0$$

so that our criterion for stability is

$$\frac{(\lambda+2\mu)}{2\mu} - \frac{(n-1)}{2n - 1} - \frac{(n^2-1)}{4n^2 - 1}\left[\frac{\lambda}{2\mu} + \frac{(\lambda + 2\mu)(1 + 4S)}{2\mu}\,U_n\right] > 0$$

or

$$(48) \qquad (1+4S)U_n > \frac{n[3n\lambda + 2(2n + 1)\mu]}{(n^2-1)(\lambda+2\mu)}$$

which is a transcendental inequality in S with terms dependent upon n and the ratio of the elastic constants. In particular, if $\mu >> \lambda$, corresponding to zero Poisson's ration, the lowest limit for S occurs for $n = \infty$ and is $S = 1/2$. For $\lambda >> \mu$, corresponding to incompressibility under hydrostatic pressure, or Poisson's ratio = 1/2, the limit for S is 1.158 and occurs for $n = 3$. In the former case, we find that $A = -1/3$ and the center of the sphere cannot close beyond half its original radius without instability. In the latter case, $A = -0.466 \frac{\mu}{\lambda}$ and instability results if the inner radius is reduced below 0.651 of its original value. We thus conclude that, in all cases, instability will result before external pressure is sufficient to halve the inner radius of the sphere.

We now turn our attention to the case of a thin sphere and let

$$(49) \qquad y = 1 - k << 1$$

We will find that S is of the order y and n^2 of the order $1/y$ for the limit of stability. Keeping this in mind to insure retention of all significant terms, we find:

$$\frac{1}{a_{n1}} \approx \frac{y}{n+2}\left\{(2n+3)[1 - (n+1)y + \frac{(n+1)(2n+1)}{3} y^2] + 6S\right\}$$

$$\frac{1}{a_{n2}} \approx \frac{y}{n-1}\left\{(2n-1)[1 + ny + \frac{n(2n + 1)}{3} y^2] - 6S\right\}$$

$$2k^2 \int_k^1 \frac{dx}{x^3} = 1 - k^2 = y(2 - y)$$

$$(2n+3)k^{2n+3} \int_k^1 \frac{dx}{x^{2n+4}} \approx (2n+3)y[1 - (n+1)y + \frac{(n+1)(2n+1)}{3} y^2]$$

$$(2n-1) \int_k^1 x^{2n-2}\, dx \approx (2n-1)y[1 + ny + \frac{n(2n + 1)}{3} y^2]$$

$$2k^2 \int_k^1 \frac{dx}{x^3(1+Sx^3)} \approx y(2 - y - 2S)$$

$$(2n+3)k^{2n+3} \int_k^1 \frac{dx}{x^{2n+4}(1+Sx^3)} \approx (2n+3)y[1-S-(n+1)y + \frac{(n+1)(2n+1)}{3}y^2$$

$$(2n-1) \int_k^1 \frac{x^{2n-2}\, dx}{1 + Sx^3} \approx (2n-1)y[1 - S + ny + \frac{n(2n+1)}{3} y^2]$$

so that

$$\beta_{n1} \approx \frac{y}{2\mu(2n+1)} \Big\{2[n\lambda+(3n+1)\mu][1 - (n+1)y + \frac{(n+1)(2n+1)}{3} y^2] + 3n(\lambda + 2\mu)S\Big\}$$

$$\beta_{n2} \approx \frac{y}{2\mu(2n+1)} \Big\{2[(n+1)\lambda + (3n+2)\mu][1 + ny + \frac{n(2n+1)}{3} y^2] + 3(n + 1)(\lambda + 2\mu)S\Big\}$$

$$\frac{\lambda+2\mu}{2\mu a_{n1}} - \beta_{n1} \approx \frac{y}{2\mu(n+2)(2n+1)} \Big\{[(2n^2+4n+3)\lambda+2(n^2+n+1)\mu][1-(n+1)y + \frac{(n+1)(2n+1)}{3} y^2] - 3(n^2 - 2n - 2)(\lambda + 2\mu)S\Big\}$$

$$\frac{\lambda+2\mu}{2\mu a_{n2}} - \beta_{n2} \approx \frac{y}{2\mu(n-1)(2n+1)} \Big\{[(2n^2+1)\lambda+2(n^2+n+1)\mu][1+ny + \frac{n(2n+1)}{3} y^2] - 3(n^2 + 4n + 1)(\lambda + 2\mu)S\Big\}$$

$$\frac{\alpha_{n1}\beta_{n1}}{\alpha_{n2}} \approx \frac{y}{2\mu(n-1)(2n+1)} \left\{ \left[\frac{2n(n+2)(2n-1)}{2n+3}\lambda + \frac{2(n+2)(2n-1)(3n+1)}{2n+3}\mu \right] \right.$$

$$\cdot \left[1 + ny + \frac{n(5n+1)}{3} y^2\right]$$

$$\left. + \frac{3(n+2)S}{(2n+3)^2} \left[n(4n^2-12n-11)\lambda + 2(4n^3-20n^2-23n-4)\mu\right] \right\}$$

$$> \frac{\lambda+2\mu}{2\mu\alpha_{n2}} - \beta_{n2} \qquad \text{in view of the order of } n^2$$

so that we need only consider criterion (46) which reduces to

$$3(3\lambda+2\mu)(\lambda+2\mu)(1-y) + [(n^4+2n^3+2n^2+n+3)\lambda^2$$

$$+ (n^4+2n^3+3n^2+2n+4)\mu(2\lambda+\mu)]y^2 - 9(\lambda+2\mu)[(2n^2+2n+1)\lambda+2(n^2+n-1)\mu]S > 0$$

or, dropping y compared to 1, and higher powers of n^2,

$$S < \frac{3\lambda + 2\mu}{6(\lambda+\mu)N} + \frac{(\lambda+\mu)Ny^2}{18(\lambda + 2\mu)}$$

where

$$N = n(n + 1)$$

This is a minimum for

$$N = \frac{\sqrt{3(\lambda+2\mu)(3\lambda+2\mu)}}{(\lambda+\mu)y}$$

for which value of N, it becomes

$$S < \frac{1}{3}\, y \sqrt{\frac{3\lambda+2\mu}{3(\lambda+2\mu)}}$$

implying that

$$A > -\frac{4\mu y}{3\sqrt{3(\lambda+2\mu)(3\lambda+2\mu)}}$$

and

$$p\ (\text{critical}) = \frac{4\mu}{3}\, y \sqrt{\frac{3\lambda+2\mu}{3(\lambda+2\mu)}}\ (1-k^3)$$

$$\approx 4\mu y^2 \sqrt{\frac{3\lambda+2\mu}{3(\lambda+2\mu)}}$$

$$= 4\mu(1-k)^2 \sqrt{\frac{3\lambda+2\mu}{3(\lambda+2\mu)}}$$

which agrees with results obtained by other methods.

Appendix A

When an integral is minimized by the application of Calculus of Variations to functions in the integrand, the value of the integral for the minimizing functions can be obtained in a simpler way than by direct substitution and integration, particularly when the integrand is a homogeneous quadratic expression in the functions and their derivatives. Consider, to be specific, a double integral involving two functions in this way. The general form is:

$$W = \int_d^c \int_b^a Q(F, F_x, F_y, G, G_x, G_y)\, dxdy$$

This is a minimum when

$$\frac{\partial Q}{\partial F} = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial F_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial F_y}\right)$$

and

$$\frac{\partial Q}{\partial G} = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial G_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial G_y}\right)$$

Now Q is a homogeneous quadratic expression, hence it may be written as

$$Q = \frac{1}{2}\left(F\frac{\partial Q}{\partial F} + F_x\frac{\partial Q}{\partial F_x} + F_y\frac{\partial Q}{\partial F_y} + G\frac{\partial Q}{\partial G} + G_x\frac{\partial Q}{\partial G_x} + G_y\frac{\partial Q}{\partial G_y}\right)$$

Now, integrating by parts:

$$\int_d^c F_x\frac{\partial Q}{\partial F_x}\,dx = F\frac{\partial Q}{\partial F_x}\Big|_d^c - \int_d^c F\frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial F_x}\right)dx$$

$$\int_d^c G_x\frac{\partial Q}{\partial G_x}\,dx = G\frac{\partial Q}{\partial G_x}\Big|_d^c - \int_d^c G\frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial G_x}\right)dx$$

$$\int_b^a F_y\frac{\partial Q}{\partial F_y}\,dy = F\frac{\partial Q}{\partial F_y}\Big|_b^a - \int_b^a F\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial F_y}\right)dy$$

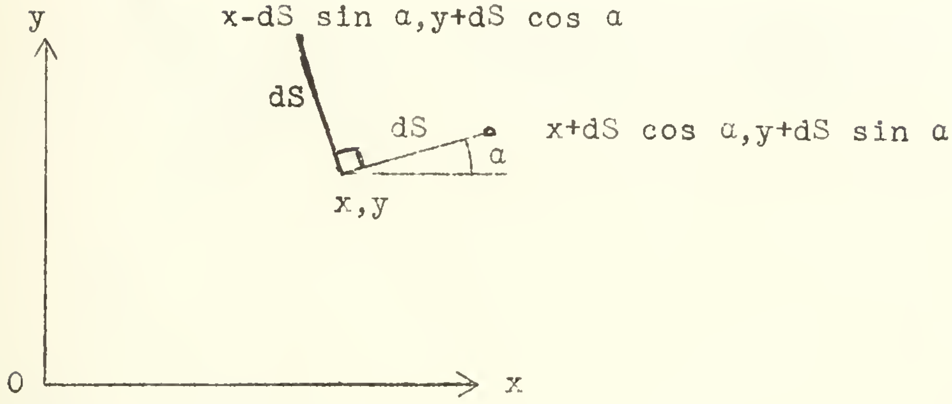
$$\int_b^a G_y\frac{\partial Q}{\partial G_y}\,dy = G\frac{\partial Q}{\partial G_y}\Big|_b^a - \int_b^a G\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial G_y}\right)dy$$

so that, in view of the minimizing equations,

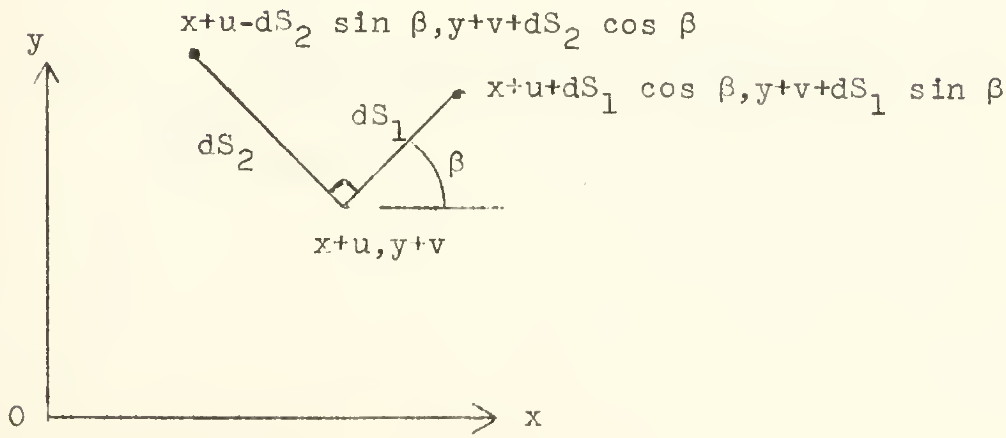
$$W = \frac{1}{2}\int_b^a \left[F\frac{\partial Q}{\partial F_x} + G\frac{\partial Q}{\partial G_x}\right]\Bigg|_d^c dy + \frac{1}{2}\int_d^c \left[F\frac{\partial Q}{\partial F_y} + G\frac{\partial Q}{\partial G_y}\right]\Bigg|_b^a dx$$

Strain Along Principal Directions

Cartesian Coordinates



Unstrained

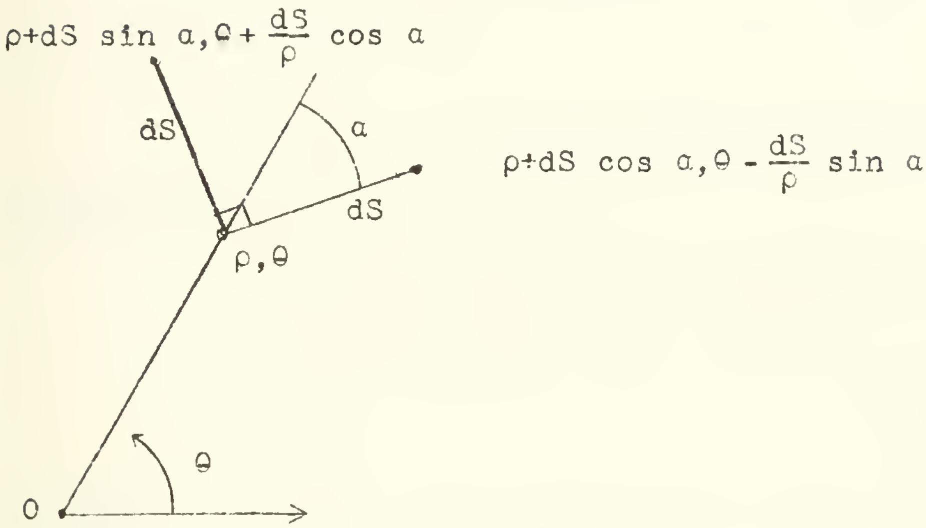


Strained

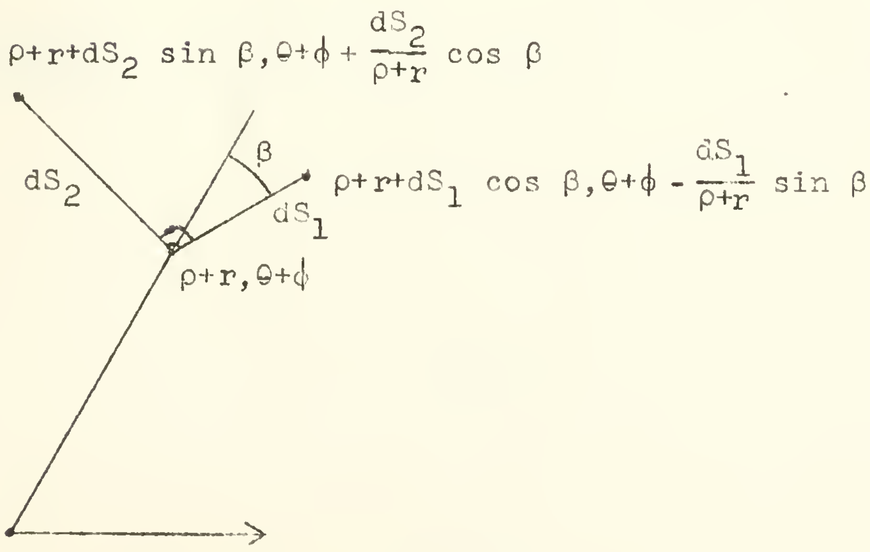
Fig. 1.

Strain Along Principal Directions

Polar Coordinates



Unstrained



Strained

Fig. 2.

Buckling of a Column

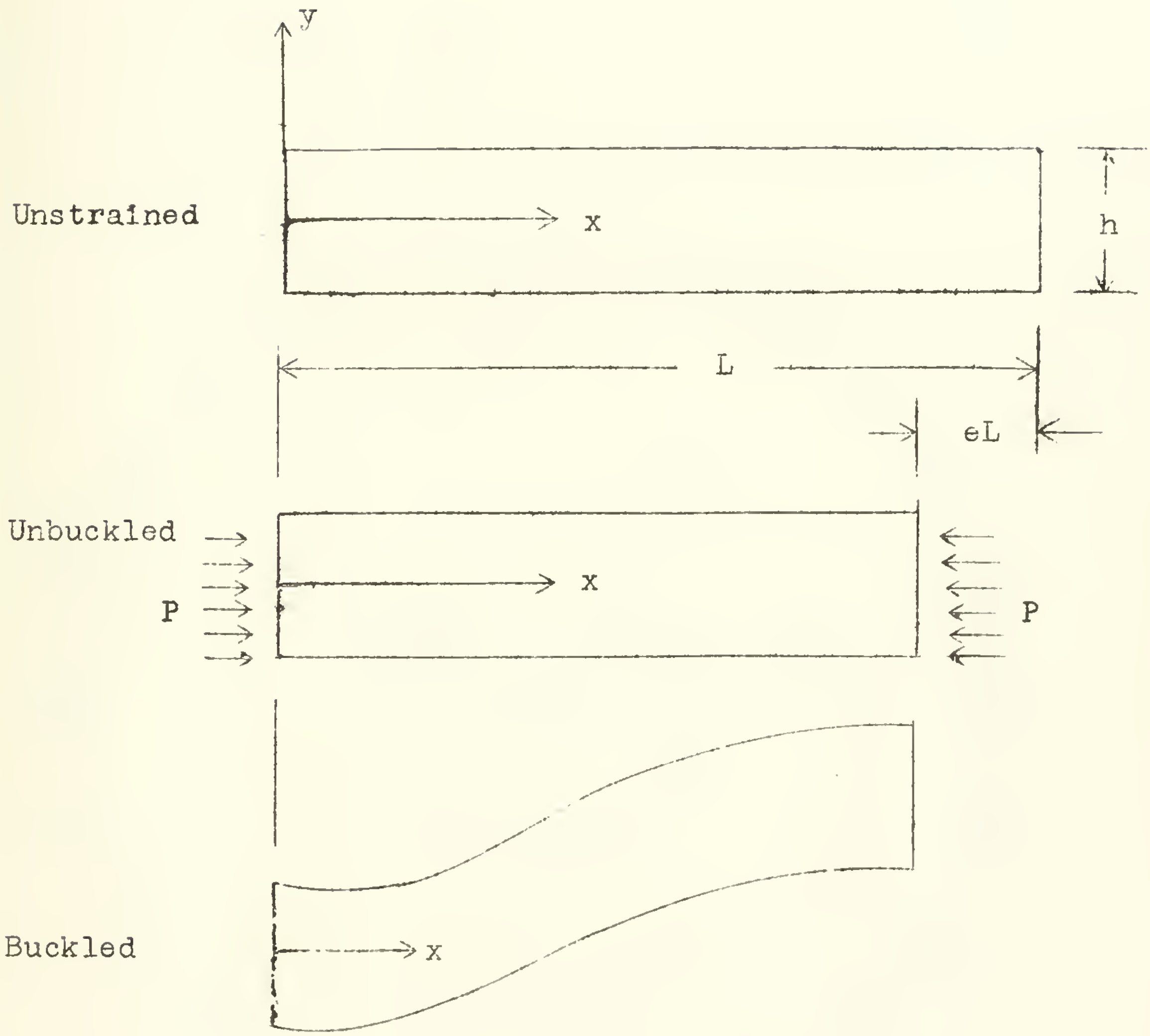
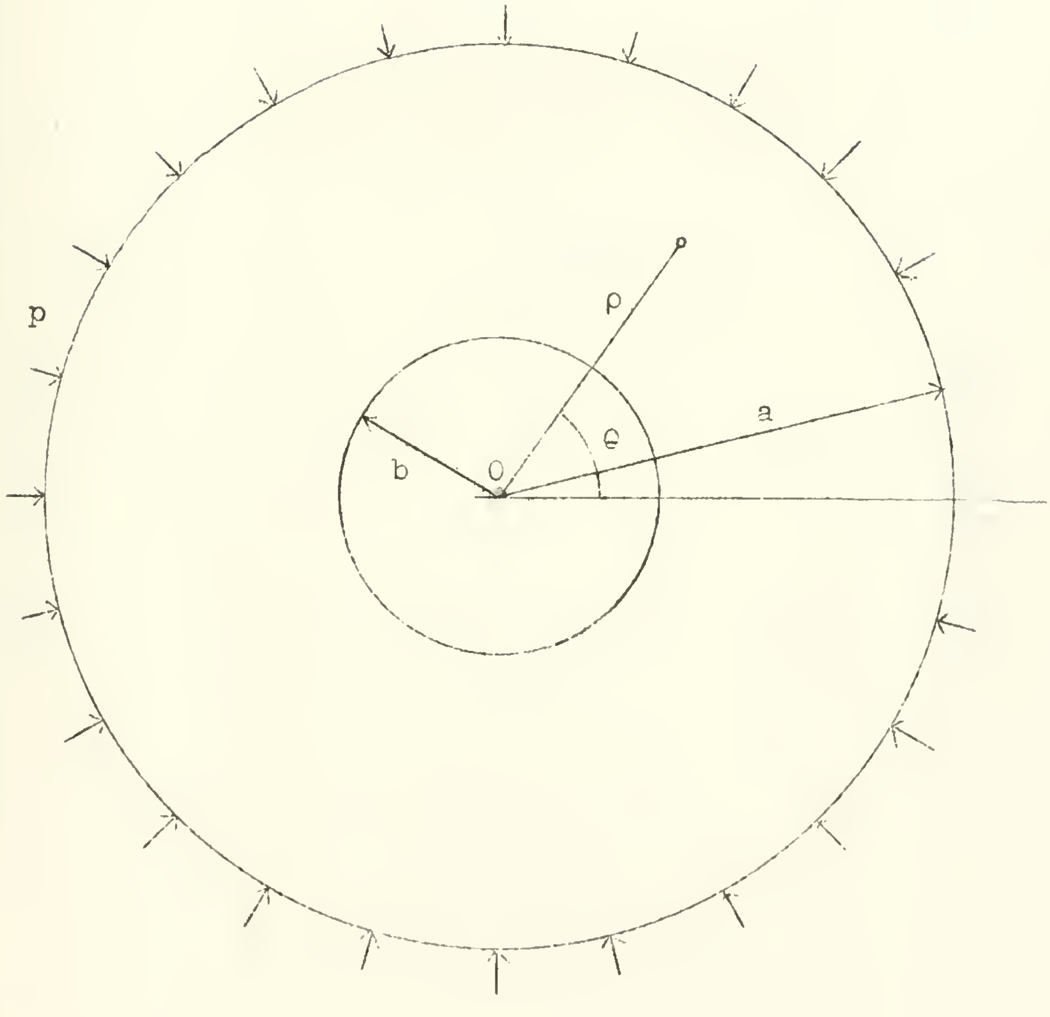


Fig. 3.

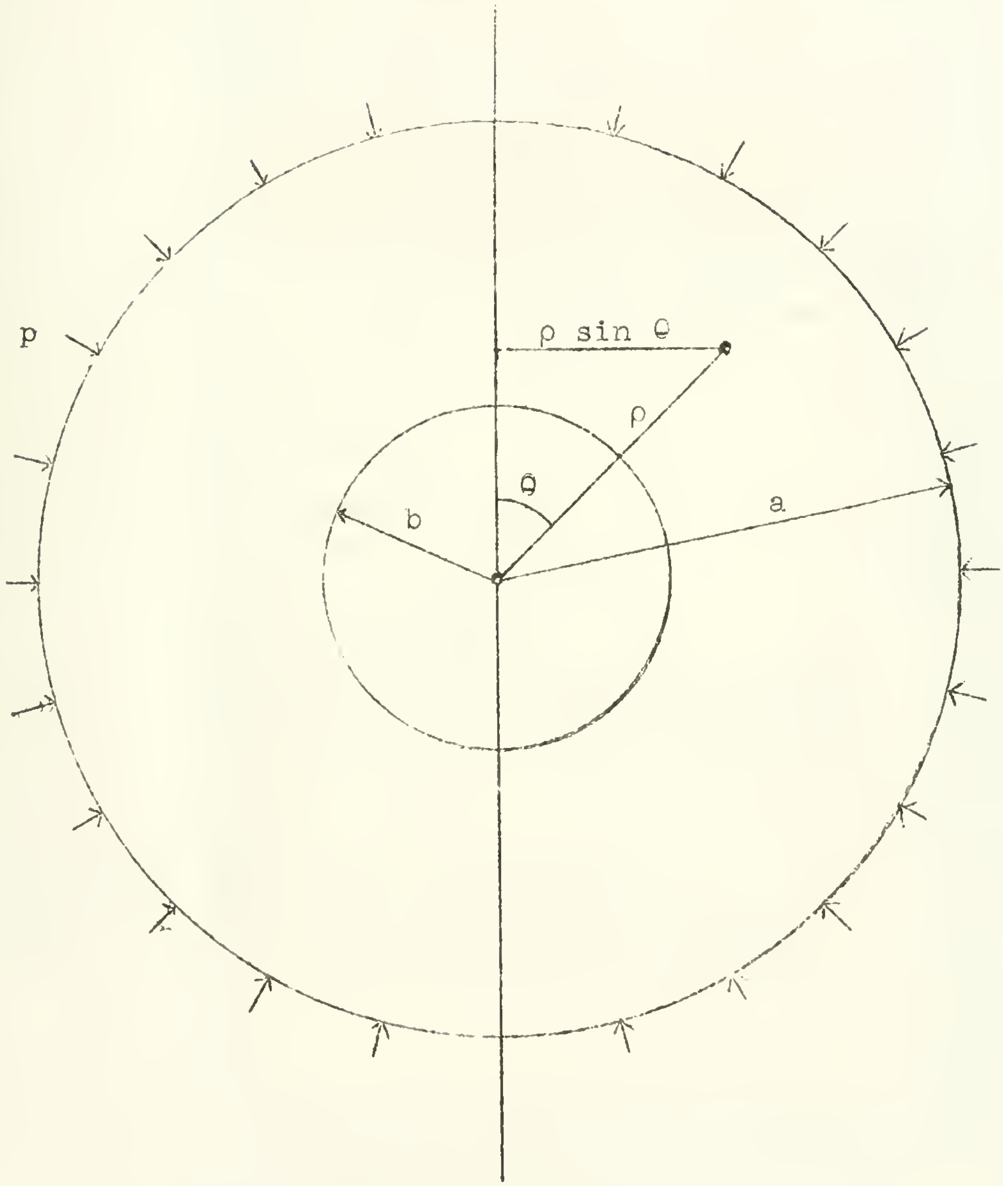
The Hollow Cylinder



Initial Condition

Fig. 4.

The Hollow Sphere



Axis of Symmetry

Initial Condition

Fig. 5.

Symbols

$\approx$	approximates to the specified order
λ, μ	Lame elastic constants
E	Young's Modulus
a,b	Radii of cylinder and sphere surfaces
k	Ratio b/a
L	Length of column
h	Thickness of column
W	Total strain energy
w	Strain energy density per unit volume
V	Total work done against external forces
T	Total change in energy of system
P	Total force
p	Pressure per unit area
e	Unit strain
dS	Incremential length
n,m	Integers
x,y	Cartesian coordinates
ρ, θ	Polar coordinates
u,v	Displacements in x and y directions
r, ϕ	Displacements in ρ and θ directions
α, β	Angles
$\triangle W$	Change in W
F, G, ψ, f, ξ	Functions of coordinates
θ, η, x	Auxiliary independent variables